Finite Groups, Modular Representations and the Green Correspondence.

by Terence Collins

Ph.D. thesis

University of Warwick

May 1973

CONTENTS

Chapter 1 Introduction and Synopsis

This thesis is mostly concerned with the block theory of finite groups whose 2-sylow subgroups are assumed to be elementary abelian of order 4. In chapter 3 indecomposable representations of the cyclic group of order 2 over a 2-adic ring of characteristic 0 are classified, and in chapter 4 this is used to describe the Green correspondence and the modular constitution of modules in 2-blocks of G of defect 1.
Chapter 5 describes the decomposition matrix of the principal block, and the Green correspondence is shown to be of a fairly simple form. The submodule lattices of projective indecomposables are discussed in chapter 6 and are used to determine the loewy and composition factors of the kernels of a projective resolution of k_G. The groups PSL(2,q), q =3,5(mod 8) satisfy the assuptions made about our group G, and in chapter 7 these groups are shown to satisfy our results.
Some wider topics are discussed in chapter 8. The methods used in the body of the thesis are shown to be applicable outside the main class of groups considered.

Chapter 2 Notation and Preliminaries.

Let G be a finite group and k a field. Throughout this thesis "kG-module" will mean a vector space of finite dimension over k on which G acts as a group of linear transformations. Usually all operators will act on the right. If R is a ring, an RG-module will be a free R-module of finite rank over R on which G acts.

If K is an algebraic number field, and P is a prime ideal in the ring I of integers in K, the P-adic valuation $v_P: K \to Z$ will be that map such that $aI = P^{v_P(a)}.A$ where A and P are coprime.

Then $v_P(ab) = v_P(a)+v_P(b)$ and $v_P(r+s) \geqslant \min[v_P(r), v_P(s)]$

The subset R of K of those elements r with $v_P(r) \geqslant 0$ is thus a subring of K. It contains I and is called the ring of P-local integers of K.

The subset $P' = \{r, v_P(r) > 0\}$ is the unique prime ideal of R and all non-zero ideals are powers of P'. R is a principal ideal domain with say $P' = \pi R$. R/P' is a finite field of characteristic p where $pZ = Z \cap P$.

We shall have occasion to use such "p-adic" rings. Further details may be found in for example C.R.

Let H be a subgroup of G : $H \leqslant G$, and let M be a kH module. M^G will denote the induced module : $M \otimes_{kH} kG$.

If N is a kG module, $N|_H$ will denote the kH module obtained by restricting the operators on N to kH. Let $K \leqslant G$. Then we have

(2.1) (Mackey) $M^G|_K \cong \oplus\sum_{\Gamma} M^g|_{H^g \cap K}{}^K$ where Γ is a set of double coset representatives of H,K in G: $G = H.\Gamma.K$ and M^g is the H^g module on which h^g acts just as h acts on M.

Choosing 1 in Γ, and H=K we get that M is a summand in $M^G|_H$.

k_G will denote the module of rank 1 over k on which G acts as the identity.

If G has a permutation representation $t:G\to S_\Omega$ on some set Ω , then the set $k\Omega$ of formal k-combinations of elements of Ω is a kG module under the action $\left(\sum k\omega\right)g = \sum k(\omega t(g))$

With respect to the basis Ω of this module , elements of G are represented by permutation matrices, which are orthogonal:i.e. their inverse and transpose coincide. It follows that any such module is self contragredient. The module $k_H^{\,G}$ is just the module obtained in this way from the permutation representation of G on the cosets of H.

In general M^* will denote the contragredient module. It is true that if M,N are kH, kG modules respectively, then $M^{*G}\cong M^{G*}$, and $N^*_{\,G}\cong N_G^{\,*}$

Similarly if σ is a ring endomorphism of k, M^σ will denote the kG representation obtained by defining $M^\sigma(g) = M(g)^\sigma$ where M(g) is the k-matrix representing g on M. The module M^σ is independent of the choice of basis of M.

As above, this operator on modules commutes with induction and restriction. Let R, P' be the p-local ring and its unique maximal ideal discussed above. In a similar way , RG-modules M can be made into R/P'-modules by applying the ring epimorphism $R\xrightarrow{\pi}R/P'\longrightarrow 0$. Similar remarks apply when R=R*, and M is an R*G-module. We write in this case $M^\pi = \overline{M}$.

R* is the completion of R with respect to its valuation.

We shall use the following result (see Thompson (15))

(2.2) If M is an R^*G module and M has no projective summands, then $\overline{M}$ has no projective summands. We say M (and $\overline{M}$) is"projective-free".

Let k be a field of finite characteristic with automorphism σ and let k_σ be the fixed field of σ . Then if F is an absolutely irreducible kG module we have (Brauer (16))

(2.3) $F^\sigma\cong F \Rightarrow$ there exists a $k_\sigma G$ module E such that $E^k = F$.

Here E^k means the kG module $E\otimes_{k_\sigma}k$

We shall say in this case that F is realisable in k_σ.

The symbol $M\mid N$ will mean that some summand of N is isomorphic to M.

If k is a field and M is an indecomposable kG module there exists in G a subgroup D with the properties

(1) there is a kD module S such that $M \mid S^G$. We say that M is D-projective.

(2) if M is C-projective, $C \leqslant G$, then a conjugate of D is contained in C

We write $D \leqslant_G C$. The subgroup D is clearly unique up to conjugacy in G and is called the vertex of M. If k has characteristic p then D is a p subgroup. If D has index p^a in some p-sylow subgroup of G then $p^a \mid |M| = \dim(M)$. We shall write $vx(M)$ for vertex(M).

These results are to be found in Green (8).

(2.4) Green Correspondence (Green (9))

Let $D \leqslant N_G(D) \leqslant H \leqslant G$. Let $\mathcal{X} = \{D \cap D^g, g \notin H\}$ and $\mathcal{Y} = \{H \cap D^g, g \notin H\}$

Let M,N be indecomposable kG,kH modules respectively, with vx(M) and vx(N) in D but not contained in any conjugate of any subgroup X in $\mathcal{X}$.

Then there exists a kG-module g(N), and kH-module f(M), both indecomposable, with

$$M|_H \cong f(M) \oplus M_{\mathcal{Y}} \qquad N^G \cong g(N) \oplus N_{\mathcal{X}}$$
$$f(M)^G \cong M \oplus f(M)_{\mathcal{X}} \qquad g(N)|_H \cong N \oplus g(N)_{\mathcal{Y}}$$

where the modules $M_{\mathcal{Y}}$,$g(N)_{\mathcal{Y}}$ are sums of modules each of which is Y-projective for some Y in $\mathcal{Y}$, and the modules $N_{\mathcal{X}}$, $f(M)_{\mathcal{X}}$ are sums of modules each of which is X-projective for some X in $\mathcal{X}$. In addition, $vx(f(M)) = vx(M)$,$vx(g(N)) = vx(N)$ and clearly, $g(f(M)) \cong M$,$f(g(N)) \cong N$.

In applying 2.4 , given an indecomposable kG-module M, f(M) will be taken as a specific kH-module satisfying the conditions above. Since f(M) is only determined up to isomorphism ,the choice of f(M) is to this extent arbitrary. A similar convention will apply to g(N).

Now let $Y \leqslant Z$ be finite groups and Γ a transversal of Y in Z : $Z = Y.\Gamma$. Let A , A' be kZ-modules. (A,A') will denote k-linear maps $A \to A'$, $(A,A')_Z$ will denote the set of kZ-homomorphisms $A \to A'$, and $(A,A')_Y$ the set of kY-homomorphisms $A|_Y \to A'|_Y$ .Let $\rho \in (A,A')_Y$ Define $T_{Y,Z}(\rho): A \to A'$ by $T_{Y,Z}(\rho): a \to \sum_{g \in \Gamma} [\rho(ag^{-1})]g$. Then $T_{Y,Z}(\rho) \in (A,A')_Z$. The set $\{T_{Y,Z}(\rho) : \rho \in (A,A')_Y\}$ is a subgroup of $(A,A')_Z$ denoted $(A,A')_{Y,Z}$. It is the subgroup of D-projective maps.

Green has also shown (Green (17))

$$(2.5)\quad \frac{(M,M')_{D,G}}{(M,M')_{\mathfrak{X},G}} \cong \frac{(f(M),f(M'))_{D,H}}{(f(M),f(M'))_{\mathfrak{X},H}}$$

where $(Z,Y)_{\mathfrak{X},G}$ means $\sum_{X\in\mathfrak{X}} (Z,Y)_{X,H}$

It is not difficult to see that (1)-projective maps are precisely those that can be factored through a projective presentation of M'. Thus if $P \xrightarrow{\pi} M' \longrightarrow 0$ is exact with P projective, the set of maps of the form $M \longrightarrow P \xrightarrow{\pi} M'$ are the projective ones i.e. $(M,M')_{1,G}$

<u>(2.6) If M' is irreducible and M is projective free then</u> $(M,M')_{1,G} = 0$

<u>Proof</u>

Since M' is irreducible, a minimal projective presentation of M' $P \xrightarrow{\pi} M' \longrightarrow 0$ has $\phi(P)$ in its kernel. $\phi(P)$ = the intersection of the maximal submodules of P = in this case the unique maximal submodule of P. (see C.R. 54.11 and 54.13)

Now let $f:M \to M'$ be projective. Then f can be factored through π viz. $f: M \xrightarrow[f']{} P \xrightarrow{\pi} M'$ Now f' cannot be surjective since then it would split (P is projective) and M would not be projective free. Hence $\mathrm{im}(f') \leqslant \phi(P)$ $\phi(P) = \ker(\pi)$ and so $f'\pi = 0$ i.e. $f = 0$ as required.

We shall use the properties of the Green correspondence without always mentioning it explicitly. 2.4 is also valid when k = R* the completion of R with respect to the P-valuation. As before let $\overline{M}$ denote the R*/PR* module obtained from the R* module by taking coefficients mod(PR*) Suppose $\overline{M}$ is indecomposable and $vx(\overline{M})$ satisfies the condition of 2.4.

<u>(2.7) If $\mathfrak{X} = \mathcal{Y} = \{1\}$, then $\overline{f(M)} = f(\overline{M})$. Similarly if N is an R*H module and $vx(\overline{N})$ satisfies 2.4 then $g(\overline{N}) = \overline{g(N)}$</u>

<u>Proof</u>

Consider the second case. $N^G = g(N) \oplus N_{\mathfrak{X}}$ $\mathfrak{X} = \{1\} \implies$ that $N_{\mathfrak{X}}$ is projective. Likewise $\overline{N}^G = g(\overline{N}) \oplus \overline{N}_{\mathfrak{X}}$ with $\overline{N}_{\mathfrak{X}}$ projective.

Now $\overline{N^G} = \overline{N}^G$ and so $\overline{g(N)} = g(\overline{N}) \oplus N'$ with $N' \mid \overline{N}$ and so $N'$is projective 2.2 now implies that $N' = 0$ and so $\overline{g(N)} = g(\overline{N})$ as required.

If U is projective when restricted to $X \in \mathfrak{X}$ we shall say that U is projective on X (and $\mathfrak{X}$*).It is then a consequence of 2.1 that the $\mathfrak{X}$- and $\mathcal{Y}$ -projective summands of the correspondent of U (see 2.4) are in fact projective. So if U is for example an R*H module and $\overline{U}$ satisfies 2.4 then we can apply the argument of 2.7 and obtain $g(\overline{U}) = \overline{g(U)}$.

In addition a kH module N is projective on $\mathfrak{X}$ if and only if $g(N)$ is projective on $\mathfrak{X}$. This is a simple consequence of 2.4 and 2.1.

(2.8) Any extension $0 \to M \to E \to N \to 0$ with M projective on D, and N D-projective, splits.

Proof

Since $M|_D$ is projective the extension splits over D, and then since N is D-projective it splits over G.(see Higman (13)).

A similar result holds with M and N reversed.

*Here we mean that U is projective on $\mathfrak{X} \Leftrightarrow \forall X$ in $\mathfrak{X}$, $U|_X$ is projective.

Also if M or M' is projective on $\mathfrak{X}$ then $(M,M')_{\mathfrak{X},G} = (M,M')_{\{1\},G}$ for, any X map where $X \in \mathfrak{X}$ is projective as M (or M') is projective on X and so any $\mathfrak{X}$ -projective map is 1-projective.

These elementary properties of modules projective on $\mathfrak{X}$ will be fairly useful in what follows.

Blocks

The representations of a group G find themselves distributed into 'blocks' If k has characteristic p the p-block containing k_G will be denoted B_0 or $B_0(G)$ or even more precisely $p\text{-}B_0(G)$. This is the principal block. Let D,H, etc. be as in 2.4. Let B be any block of G. Suppose that the defect group of B $\leq D$ but is not contained in any conjugate of any subgroup X of $\mathscr{X}$. Then there exists a block $b = f(B)$ of kH whose defect group equals that of B. This (Brauer) correspondence between blocks of such defect groups is one to one. If M is a kG indecomposable satisfying 2.4 then Green has proved

<u>(2.9) $M \in B \Leftrightarrow f(M) \in f(B)$</u>

In the case where D is sylow in G $f(B_0(G)) = B_0(H)$

We shall now mention some general properties of modules and projective presentations.

Let η = nilpotent radical of kG, k of characteristic p. If M is a kG module, for some n $M\eta^n = 0 \neq M\eta^{n-1}$. n is called the loewy length of M. It is the shortest length of a submodule series of M which has completely reducible factors.

Let s(M) = the sum of all irreducible submodules of M. We shall call s(M) the socle or foot of M. $\phi(M)$ = the intersection of all maximal submodules of M. $\phi(M)$ is the frattini submodule of M and $M/\phi(M)$ will be referred to as its head. We shall call M single footed (headed) in the case that the foot (head) is irreducible.

Suppose M has no irreducible summands. Then $s(M) \leq \phi(M)$ since any irreducible submodule I not contained in some maximal submodule S must satisfy $I \cap S = 0$ and $I + S = M$ and so $I \oplus S = M$ contradicting M irreducible-free. (i.e. M has no irreducible summands.)

$M/\phi(M)$ is completely reducible $= F_1 \oplus F_2 \dots \quad \dots F_t$ say.

Now for every i there exists a P_i such that P_i is projective indecomposable and the head of $P_i = F_i$ (see C.R. 54.13) We can, therefore construct the following diagram of exact row and column

$$\begin{tikzcd}
& & 0 & \\
M \arrow[r, "\rho"] & M/\phi(M) \arrow[r, phantom, "\cong"] & F_1 \oplus \cdots \cdots \oplus F_t \arrow[u] \arrow[r] & 0 \\
& & P_1 \oplus \cdots \cdots \oplus P_t \arrow[u, "\pi"] \arrow[ull, dashed, "\pi'"] &
\end{tikzcd}$$

Since $\oplus P_i$ is projective we can lift π to M. Thus we have the map π' of the diagram. $\pi'\rho$ is an epimorphism ($=\pi$) and so $\mathrm{im}(\pi') + \ker(\rho) = M$ However as $\ker(\rho) = \phi(M)$ this means that $\mathrm{im}(\pi') = M$ and so π' is a projective presentation of M. $\ker(\pi') \leq \ker(\pi) = \phi(P_1+P_2+\ \ +P_t)$ Such a presentation is called minimal. It is minimal in the sense that no projective presentation of M can have fewer than t indecomposable summands. The kernel of π' will be denoted $\Omega(M)$. It is projective free and is indecomposable if and only if M is. The operator Ω has an inverse, obtained by taking the cokernel of a minimal injective embedding of any module e.g. $\Omega(M)$.

Let $0 \to \Omega(M) \to P \to M \to 0$ be a minimal presentation of the projective-free module M. $P = P_1 \oplus P_2 \oplus P_3 \oplus \quad \oplus P_t$ say. Then clearly

$s(P) = \oplus\, s(P_i)$

Suppose $s(P) \nleq \Omega(M)$. Then for some i, $s(P_i) \cap \Omega(M) < s(P_i)$

i.e. $s(P_i) \cap \Omega(M) = 0$ and so $P_i \cap \Omega(M) = 0$

Since $\dfrac{P_i + \Omega(M)}{\Omega(M)} \cong \dfrac{P_i}{P_i \cap \Omega(M)} \simeq P_i$ it follows that P_i

occurs as a submodule of (M). Since (M) is projective-free this cannot be the case. Thus we must have $s(P) \leq \Omega(M)$.

$\Omega(M) \leq P$ now implies that $s(P) = s(\Omega(M))$. However, by the construction of the minimal presentation $s(P) \cong P/\phi(P) \cong M/\phi(M)$.

Thus we have proved $s(\Omega(M)) \cong M/\phi(M)$.

This argument also shows that in any minimal projective presentation $P \xrightarrow{\pi} M \to 0$, $s(P) \leq \ker(\pi)$, and so the loewy length of M is less than that of P. From this it is not difficult to see that the maximal loewy length of the projective modules in a block is one more than the maximum for projective free modules.

<u>(2.10) Let M have a series of submodules $M = M_0 \supset M_1 \supset M_2 \supset \quad M_t = 0$ with factors $F_1\ F_2 \quad F_t$ and suppose that F_i say, is projective. Then $M \cong F_i \oplus M'$ and M' has a submodule series with factors $F_1\ F_2\ \widehat{F_i}\ \ F_t$</u>

<u>Proof</u>

Consider $\dfrac{M_{i-2}}{M_i}$. This has a submodule $\dfrac{M_{i-1}}{M_i} \cong F_i$

and the quotient $\dfrac{M_{i-2}}{M_{i-1}} \cong F_{i-1}$

F_i is projective and so we may write $\dfrac{M_{i-2}}{M_i} \cong F_i \oplus F_{i-1}$

and there exists a submodule say M'_{i-1} which may replace M_{i-1} in the original series but such that M_{i-2}/M'_{i-1} is now projective. Repeating this argument we obtain eventually the form stated in 2.10

A similar argument applies when the factor F_i is in a block not shared by any other F_j

<u>(2.11) The modules $\{\Omega^{\nu}(k_G), \nu \in \mathbb{Z}\}$ k a modular field, are realisable in $GF(p) \subseteq k$. The modules $\{\Omega^{\nu}(R_G), \nu \in \mathbb{Z}\}$, R a p-adic ring are realisable in $\mathbb{Z} \subseteq R$. The characters $\chi(\Omega^{\nu}(R_G)), \chi(\Omega^{-\nu}(R_G))$ are equal and rational.</u>

<u>Proof</u>

The operator Ω commutes with extension of the base field (or ring) and Ω and Ω^{-1} are additive. Since k_G and $\mathbb{Z}_G$ are absolutely indecomposable this has the implication that $\Omega^{\nu}(GF(p)), \Omega^{\nu}(\mathbb{Z}_G)$ are absolutely indecomposable. Extending $GF(p)$ and $\mathbb{Z}$ to k and R respectively now yields the first part. It also implies that $\chi(\Omega^{\nu}(R_G))$ are rational.

Now apply $*$ to the exact series $0 \to \Omega(M) \to P \to M \to 0$

We get an exact series $0 \to M^* \to P^* \to \Omega(M)^* \to 0$. Since P^* is projective this means that $\Omega(\Omega(M)^*) \cong M^*$ This yields, on application of Ω^{-1} that $\Omega^{-1}(M^*) = (\Omega(M))^*$

It follows that the modules $\Omega^{\nu}(\mathbb{Z}_G)$, $\Omega^{-\nu}(\mathbb{Z}_G)$ are contragredient and their characters are complex conjugate. The fact that these characters are rational means that they must be equal . This proves all parts of 2.11

M^* is the module of k-maps (R-maps) $M \to k$ with action $fg(m) = f(mg^{-1})$ where $m \in M$ and $g \in G$. If N is any submodule of M the set of f $f: M \to k$ with $f(N) = 0$ is a submodule of M^* . This defines a bijection between the submodule lattices of M and M^* .Call it T. Then * inverts the submodule lattice of M in the following sense :

$N \subseteq N'$ if and only if $T(N) \supseteq T(N')$,

and $N'/N \cong (T(N)/T(N'))^*$. That is, corresponding sections are contragredient.

If u is an automorphism of the field k, the modules M and M^u have identical submodule lattices and corresponding sections are related by $(N'/N)^u \cong N'^u/N^u$

The module operators Ω , * , u , and extension of the base field all commute with the Green correspondences f and g, preserve indecomposability,and vertices. This is because restriction and induction of modules are exact. Thus if $0 \to \Omega(M) \to P \to M \to 0$ is exact with P projective then we can restrict and induce this to the exact sequence $0 \to \Omega(M)_D{}^G \to P_D{}^G \to M_D{}^G \to 0$ with $P_D{}^G$ projective. Now if M is D-projective we have $M \mid M_D{}^G$ and Schanuel's lemma says that $\Omega(M) \mid \Omega(M)_D{}^G$ i.e. that $\Omega(M)$ is D-projective. Similarly $\Omega(M)$ D-projective implies that M is. These two facts now yield that Ω preserves vertices.

<u>(2.12) Let k be a field. Let a kG module L have series $L \supset M \supset N \supset 0$ and suppose that $N \mid M$ and $M/N \mid L/N$. Then L decomposes as $L \cong L' \oplus M/N$ and L' has a series with factors L/M , N.</u>

<u>Proof</u>

Let $n_1\ n_2\ n_3 \cdots n_r$ be a basis for N and $m_1\ m_2\ m_3 \cdots m_s$ a basis of any complement of N in M. Thus $M = (n_1\ n_2 \cdots n_r, m_1\ m_2 \cdots m_s)$ Now choose $l_1\ l_2 \cdots l_t$ so that all these elements form a basis of L and modulo the submodule $(n_1\, n_2 \cdots n_r)$, $\bar{l}_1\, \bar{l}_2 \cdots \bar{l}_t$ form a basis of a complement of $(\bar{m}_1\, \bar{m}_2 \cdots \bar{m}_s)$. With respect to the basis of L comprising all these elements, and writing operators on the left, an element g of G is represented by a matrix of the form

$$L(g) = \left[\begin{array}{c|c|c} N(g) & 0 & * \\ \hline 0 & \frac{M}{N}(g) & 0 \\ \hline 0 & 0 & \frac{L}{M}(g) \end{array}\right]$$

The submatrix * here is left undetermined. It is clear that L decomposes into M/N and an extension of N by L/M . This proves the result.

<u>(2.13) Every kG module in B_0 has $O_{p'}(G)$ in its kernel.</u>

<u>Proof</u>

Brauer(1) in Theorem 1 shows that every ordinary character in B_0 has $O_{p'}(G)$ in its kernel. Since every modular irreducible occurs as a modular constituent of some ordinary irreducible the modular irreducibles in B_0 must also have $O_{p'}(G)$ in their kernel. If M is any kG module in the principal block it has composition series $\{M_i\}$ and $O_{p'}$ acts trivially on each factor. Let $g \in O_{p'}$ The matrix $M(g) - I$ is therefore upper zero-triangular and so nilpotent. Choose n big enough such that $(M(g)-I)^{p^n} = 0$. Then since we are in characteristic p this means that $M(g^{p^n}) = I$. The fact that g is a p'-element now forces that $M(g) = I$ i.e. g is in the kernel of M. Hence $O_{p'}(G) \leq \ker(M)$.

For the remainder of this thesis, the "field k of characteristic 2" will be the algebraic closure of GF(2), unless otherwise stated. I will be a 2-adic valuation subring of $\mathbb{C}$, with residue class field $k \cong I/P$ and completion R . This is possible (see Thm. 1 Maclane (18)).

(2.14) Representation theory of A_4

Let $A_4 \cong N = \langle a,b,g\ ;\ a^2 = b^2 = (ab)^2 = 1,\ a^g = g^{-1}ag = b,\ b^g = ab, g^3 = 1\rangle$

This is the alternating group on four letters.

Its conjugacy classes are represented by $1, a\ g\ g^2$ and its character table is easily seen to be

		$\{1\}$	$\{a\}$	$\{g\}$	$\{g^2\}$
R_0	χ_0	1	1	1	1
R_γ	χ_γ	1	1	γ	γ^2
R_{γ^2}	χ_{γ^2}	1	1	γ^2	γ
V	χ	3	-1	0	0

The representations R_0 R_γ R_{γ^2} are modularly distinct and irreducible and the decomposition matrix (mod 2) is clearly

	k_N	k_γ	k_{γ^2}
R_0	1	0	0
R_γ	0	1	0
R_{γ^2}	0	0	1
V	1	1	1

If k is a field of characteristic 2 γ will denote ambiguously a cube root of one in k or $\mathbf{C}$. k_γ and R_γ will be the k- R- representations having $P = \{a,b\}$ in their kernel and mapping g to γ .

Of course R_{γ^i} affords the character χ_{γ^i} and $\overline{R_{\gamma^i}} \cong k_{\gamma^i}$

R_{γ^i} is the unique lifting of k_{γ^i} to an R - representation.

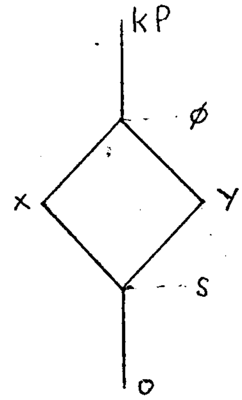
The ring $k\langle g\rangle$ has ideal decomposition $k\langle g\rangle = k\,\delta_0 \oplus k\,\delta_1 \oplus k\,\delta_2$

where $\delta_i = \sum_{j=0}^{2} \gamma^{-ij} g^j$ are orthogonal idempotents with sum 1. Embed $k\langle g\rangle$ in kN.

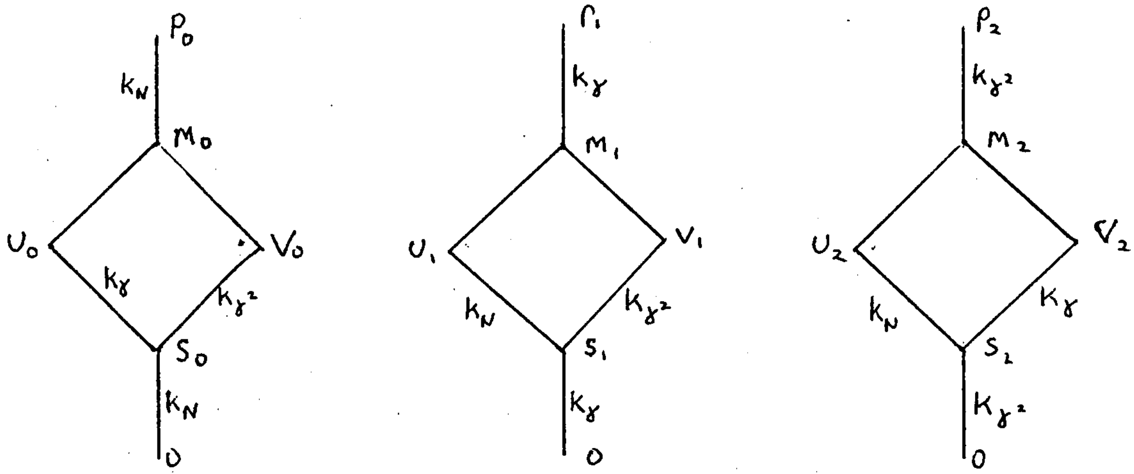
Thus we may write : $kN = \delta_0 kN \oplus \delta_1 kN \oplus \delta_2 kN$

Since $P \lhd N$ it is clear that $\delta_i kN = \delta_i kP = P_i$ say. Then P_0, P_1, P_2, are projective kN indecomposables with heads k_N k_γ k_{γ^2} respectively.

$P_i/_P \cong kP$, and kP has submodules X and Y (not unique) such that $\phi/s = X/s \oplus Y/s$.
Here $\phi = \phi(kP)$, $s = s(kP)$.
Now consider P_o, P_1 and P_2.



It is not difficult to see that the submodule lattices for these are



where this time the submodules are unique. That is, P_i has exactly six submodules.

Any indecomposable kN module of dimension 2 is clearly single-headed and so must be one of the six modules P_i/U_i, P_i/V_i

We shall denote these unambiguously by symbols

$$\left|\begin{matrix} k_{\gamma^i} \\ k_{\gamma^j} \end{matrix}\right. \qquad i, j = 0, 1, 2, \quad i \neq j$$

where this indicates the submodule lattice.

The operators Ω, $*$ and the field automorphism $\tau : k \to k : u \to u^2$ permute these modules according to the following diagram.

$$\begin{tikzcd}
P_0/V_0 \arrow[r, leftrightarrow, "\tau"] \arrow[d, dashed, "\Omega"'] \arrow[d, leftrightarrow, shift left=2, "*"] & P_0/U_0 \arrow[d, dashed, "\Omega"] \arrow[d, leftrightarrow, shift right=2, "*"'] \\
P_2/V_2 \arrow[r, leftrightarrow, "\tau"] \arrow[d, dashed, "\Omega"'] & P_1/V_1 \arrow[d, dashed, "\Omega"] \\
P_1/U_1 \arrow[r, leftrightarrow, "\tau"] \arrow[uu, dashed, bend left, "\Omega"] & P_2/U_2 \arrow[uu, dashed, bend right, "\Omega"']
\end{tikzcd}$$

These six modules are projective when restricted to a subgroup of order two.

The lattices of projective indecomposables are of the same form as those considered in chapter 5 section (1) and so the argument of theorem 6.2 can be applied. We thus obtain that $\Omega^{3r+s}(k_N)$ has head $(r+s-1)k_N \oplus (r+1)(k_\gamma \oplus k_{\gamma^2})$ and its foot is the same as the head of $\Omega^{3r+s-1}(k_N)$ (see a previous remark). Thus $\Omega^{v}(k_N)$ has dimension 2v+1 . For v negative these results are obtained by applying the functor * paying due regard to its effect on submodule lattices. (see the remarks following 2.11)

Similarly, $\Omega^{v}(k_{\gamma^i}) \cong \Omega^{v}(k_N \otimes k_{\gamma^i}) \cong \Omega^{v}(k_N) \otimes k_{\gamma^i}$ and its loewy factors are those of $\Omega^{v}(k_N)$ tensored with k_{γ^i}

At dimension 2v+1 we have thus obtained six indecomposable kN modules They are displayed on the following diagram together with the action of the two module operators τ and *

$$\begin{tikzcd}
\Omega^{v}(k_N) \arrow[rr, leftrightarrow, "*"] & & \Omega^{-v}(k_N) \\
\Omega^{v}(k_\gamma) \arrow[d, leftrightarrow, "\tau"] \arrow[rrd, leftrightarrow] & & \Omega^{-v}(k_\gamma) \arrow[d, leftrightarrow, "\tau"] \arrow[lld, leftrightarrow] \\
\Omega^{v}(k_{\gamma^2}) & {*} & \Omega^{-v}(k_{\gamma^2})
\end{tikzcd}$$

These will be shown to be the only kN-indecomposables of this dimension. Let M be any indecomposable of dimension 2v+1 . The loewy length of P_0 , P_1 , P_2 , is 3 and so M has loewy length 2. Thus $s(M) = \phi(M)$ Let s(M) and M/s(M) have dimensions s and t . s+t = 2v+1 and so $s \neq t$ Suppose first that $s > t$. Consider a minimal projective resolution of M. It has t factors and so dimension 4t. The kernel $\Omega(M)$ must then have dimension $4t-(s+t) = 3t-s$. Now $s(\Omega(M)) \cong M/\phi(M)$ and so has dimension t. dimension $\Omega(M) = 3t-s = t+s-2(s-t) < t+s = \dim(M)$. dimension $\Omega(M)$ is odd and $\dim(s(\Omega(M))) = t > 2t-s = \dim(\Omega(M)/s(\Omega(M)))$. We can , therefore repeat this argument on $\Omega(M)$ and so on. Eventually we get that $\Omega^{v}(M)$ is of dimension 1 and so is k_{γ^i} for some i. It follows that $M = \Omega^{-v}(k_{\gamma^i})$. A similar argument works in the case that $s < t$, using injective embeddings in place of projective resolutions. Thus we have proved

<u>(2.15) kN has six indecomposable representations of dimension</u> $2v+1 > 1$ <u>They are</u> $\Omega^{\pm v}(k_{\gamma^i})$ $\quad v \in \mathbb{Z},\ i = 0,1,2.$

Of dimension one kN has just the three irreducibles $k_N, k_\gamma, k_{\gamma^2}$

It is also clear now that any such module has $\dim(s(M)) = \dim(M/s(M)) \pm 1$

Now let M be any indecomposable kN module of even dimension.

<u>(2.16)</u> $\underline{\dim(s(M)) = \dim(M/s(M))}$ <u>and</u> $\underline{\dim(\Omega(M)) = \dim(M)}$.

<u>Proof</u>

Let s and t be as before ; $s+t = 2v$ this time. Consider as before a minimal projective presentation of M. It has t factors and dimension 4t So $\Omega(M)$ has dimension $4t-(s+t) = s+t-2(s-t)$. Assuming that $s > t$ the dimension of $\Omega(M)$ is seen to be less than $\dim(M)$. Also $\dim(s(\Omega(M))) = t$ and so $\dim s(\Omega(M)) > \dim(\Omega(M)/s(\Omega(M))) = t-(s-t)$. Thus we can repeat the argument with $\Omega(M)$ and eventually obtain a module which has even dimension ,frattini submodule zero,and is indecomposable. This is not possible. A similar argument in the case that $s < t$, using again injective embeddings, shows that we must set $s = t$. This proves the first part of 2.16 and the second part is an obvious consequence.

Now let (a) be a subgroup of order 2 in N. ${k_{(a)}}^N$ is indecomposable and is the unique module of kN of vertex(a). ${k_{(a)}}^P$ has a series $\left|\begin{matrix} k_P \\ k_P \end{matrix}\right.$ and ${k_P}^N = k_N \oplus k_\gamma \oplus k_{\gamma^2}$;since induction is an exact functor it is clear that ${k_{(a)}}^N$ has loewy factors

$$\begin{matrix} k_N \oplus k_\gamma \oplus k_{\gamma^2} \\ k_N \oplus k_\gamma \oplus k_{\gamma^2} \end{matrix}$$

${k_{(a)}}^N$ is the module obtained from the permutation representation of N

and so is liftable to R. ${R_{(a)}}^{N}$ and ${\bar{R}_{(a)}}^{N}$ comprise two such liftings.*
Let $A = {k_{(a)}}^{N}$. A has a unique submodule M such that $A/M \cong k_N$ and a unique submodule $I \cong k_G$. This follows from the remarks above.
Clearly $0 \subset I \subset s(A) = \phi(A) \subset M \subset A$. We shall determine M/I.
A basis for M/I is the coset sums

$$m_1 = (a).g-(a).1$$
$$m_2 = (a).g^2-(a).1$$
$$m_3 = (a).b-(a).1$$
$$m_4 = (a).bg-(a).1$$

all of these being taken modulo the submodule $I = k\sum x \quad , \quad x \in N$
With respect to this basis N is represented as follows

$$a \rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad g \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

As k contains the cube root γ of one we can choose the following basis for M/I :

$$n_1 = m_1 + \gamma m_2$$
$$n_2 = m_1 + \gamma^2 m_3 + m_4$$
$$n_3 = m_1 + \gamma^2 m_2$$
$$n_4 = m_1 + \gamma m_3 + m_4$$

With respect to this basis a and g are represented by the matrices

$$a \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ \gamma & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \gamma & 1 \end{bmatrix} \qquad g \rightarrow \begin{bmatrix} \gamma^2 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma^2 \end{bmatrix}$$

This exhibits a direct sum decomposition of M/I .The factors are easily seen to be indecomposable , and are in fact the modules $\begin{array}{c|} k_{\gamma} \\ k_{\gamma^2} \end{array}$ $\begin{array}{|c} k_{\gamma^2} \\ K_{\gamma} \end{array}$ where these have the significance of 2.14.

*See Theorem 3.1

Now let G be a group whose 2-sylow subgroup $P \cong C_2 X C_2$, $P = \{1,a,b,c\}$
$C_G(P)$ is 2-nilpotent and so $= P \times O$ say. O is of odd order and is normal in $N = N_G(P)$. $N/O \cong A_4$ By 2.13 the principal 2-block of N comprises kA_4 representations lifted mod(O) to N. By 2.9 modules in 2-$B_0(G)$ of vertex P correspond on N to modules in 2-$B_0(N)$ and so to these kA_4 modules. This is implicit in the work that follows.
Let G_γ , G_{γ^2} be the correspondents of k_γ , k_{γ^2} and $G_{\gamma^0} = k_G$ the correspondent of k_N, identifying k_{γ^i} as kN- and kA_4- module.

(2.17) Let M be any odd dimensional indecomposable kG module. M must belong to a block of defect group P. If M belongs to the principal block then it is one of the modules $\Omega^\nu(G_{\gamma^i})$.

Proof

The first statement is a consequence of remarks earlier preceding 2.4 and Green's characterisation of the defect group in Green(8). Now 2.4 and the remarks preceding it imply that $|M| \equiv |f(M)|$ (mod 2) and so f(M) is an odd dimensional module in $B_0(N)$. We can regard f(M) as an A_4 module and now 2.15 has the force that $f(M) = \Omega^\nu(k_{\gamma^i})$ for some i and ν . Since Ω commutes with the Green correspondence the result follows.

Let a be an involution in G. $C_G(a)$ is 2-nilpotent with 2-complement U say. $C = C_G(a)$ has three classes of involutions viz. $\{a\}$ conjugate only to itself, and classes $\{b\}, \{c\}$ of order $|C(a)|/|C(P)|$.

For later use we record the following result.

(2.18) $|P^x \cap C| = 4$ if and only if $x \in N.C$

In every other case $a \notin P^x$ and so $P^x \cap C$ is a C-conjugate of (b) or (c) or (1).

Proof

Suppose $|P^x \cap C| = 4$. $P^x \subseteq C$. Hence there is in C a c such that $P^{xc} = P$. i.e. $xc \in N(P)$ and so $x \in N.C$

Suppose $a \in P^x$ i.e. $a, a^{x^{-1}} \in P$. Hence for some $g \in N(P)$ $a^{x^{-1}g} = a$ i.e. $x^{-1}g \in C$. This is equivalent to $x \in N.C$

Now take $D = (a)$ in (2.4) and $H = N_G(D) = C_G(a)$.

The Green correspondence between G and C applies to modules of vertex (a)

$\mathfrak{X} = (1)$ as $(a)^x \cap (a) \neq 1$ if and only if $x \in C$.

$\mathcal{Y} = (1), (b), (c)$, together with

C-conjugates of $\{1\}, \{b\}$, or $\{c\}$. This is a consequence of 2.18.

(2.19) Let M be an indecomposable kG module in B_0 of vertex (a). Then the kC module f(M) lies in $B_0(C)$.

Proof

Let e_0 be the principal central idempotent of kG. $Me_0 = M$.

kG is a G-algebra in the sense of for example Green(10), and e_0 is primitive in kG_G. Embed this in kG_C and e_0 is not in general primitive. Let it decompose as $e_0 = f_1 + f_2 + f_m + f_{m+1} + f_n$ where $f_{m+1} \ldots f_n$ are $\mathcal{Y}$-projective and $f_1 \ldots \ldots, f_m$ have defect group containing (a), say $D_1, D_2, \ldots D_m$. Thus $C_G(D_i) \subseteq C$ and so $b_i^G = B_i$ is defined where b_i is the C-block associated with f_i. (Brauer (3) Theorem 2A). It follows from the proof of Thm. 1 (Nagao (19)) that the blocks b_i are precisely those kC blocks which satisfy $b_i^G = B_0$. Now Brauer's third main theorem (See for example Brauer(1)) says that $b_i^G = B_0$ if and only if $b_i = B_0(C)$

Thus $m = 1$ and $b_1 = B_0(C) = b_0$ say.

It follows that we can decompose e_0 in kG_C as $e_0 = f_0 + f_1 + \ldots + f_n$ with $f_1 \ldots f_n$ $\mathcal{Y}$-projective, and so over C, M decomposes as $M = Me_0 = Mf_0 \oplus (Mf_1 \ldots \oplus Mf_n)$. The bracketed module is $\mathcal{Y}$-projective and so 2.4 implies that $f(M) \mid Mf_0$ That is, f(M) belongs to $B_0(C)$, and 2.19 is proved.

2.13 implies that f(M) has $U = O_{p'}(C)$ in its kernel. f(M) has vertex (a) and so is (a) X U - projective. That is, $f(M) \mid f(M)_{(a) \times U}{}^C$. (a)X U has just two indecomposables: $k_{(a) \times U}$ and $\widetilde{k(a)}$ where this means k(a) lifted modulo U to (a)X U. This latter is projective. Hence

$f(M) \mid k_{(a)\times U}{}^{C} = \widetilde{k_{(a)}}^{P} = k_{(a)}{}^{P}$ lifted modulo U to C; this is indecomposable. Thus $f(M) = \widetilde{k_{(a)}}^{P}$.

It follows from all this that there is in $B_0(G)$ a unique module of vertex (a); we call it A'. Its correspondent on C is $\widetilde{k_{(a)}}^{P}$.

We can use these results to sharpen the Green correspondence between G and N.

(2.20) Let M be a kN module indecomposable of vertex P lying in $B_0(N)$. Then by 2.4 $M^G = g(M) \oplus \mathcal{X}$-projectives. However that part of M^G lying in B_0 $= M^G e_0 = g(M) \oplus \{1\}$-projectives.

Proof

Suppose not. Then A' is a summand in M^G. By the Green correspondence between G and C, $\widetilde{k_{(a)}}^{P}$ is a summand in $M^G{}_C$. Now apply Mackey(2.1) to this module. We have a double coset decomposition

$G = N.1.C \bigcup_x N.x.C$

2.18 has the force that $N^1 \cap C$ has 2-sylow subgroup P and all other $N^x \cap C$ have 2-sylows C-conjugate to (1), (b), or (c). Hence

$M^G{}_C = M_{N\cap C}{}^{C} \oplus \mathcal{Y}$-projectives. By our assumption we must then have $\widetilde{k_{(a)}}^{P}$ is a summand of $M_{N\cap C}{}^{C}$. $N \cap C = P \times O$ is normal in N. There exists an $N\cap C$ indecomposable of vertex P—call it M'— such that $M \mid M'^N$. Applying 2.1 to the module $M'^N{}_{N\cap C}$ we get that $M_{N\cap C}$ is a summand of $M' \oplus M'^g \oplus M'^{g^2}$ – all indecomposable of vertex P. Similarly, there exists a kP indecomposable L of vertex P with $L^{P\times O}$ containing M' as a summand. Thus to obtain a contradiction it will be sufficient to show that $\widetilde{k_{(a)}}^{P}$ is not a summand of L^C.

Firstly consider kU. U is odd and so kU is semi-simple by Maschke's theorem. Thus as right kU module kU decomposes into irreducible summands. The multiplicity of each is its dimension and so we may write

$kU = k_U \oplus \sum U_i$ where $U \nsubseteq \ker(U_i)$.

Now consider $L^C{}_U = |L| k_U \oplus \sum |L|\, U_i$

All kC modules are P-projective and so $\tilde{L} \mid \tilde{L}_P{}^C$. But $\tilde{L}_P = L$ and so $\tilde{L}$ is a summand in L^C. $\tilde{L}_U$ is trivial and so $= |L|\ k_U$. Now if L' is any other summand of L^C, L'_U must be a sum of U_i none of which is trivial on U. The only B_0-summand of L^C is $\tilde{L}$. It follows that $\widetilde{k_{(a)}}^P$ is not a summand of L^C. This contradicts our original assumption and so it must be the case that A' is not a summand of M^G. This completes the proof of 2.20. For the next result see Conlon (5).

(2.21) Let $N = A_4$. GF(2)N has just one absolutely indecomposable representation of dimension 6n over GF(2). It is, in Conlon's (5) notation $C_n(T-0)^N$. For other even dimensions there are no such modules. There are six absolutely indecomposable representations of N over GF(4) which are not realisable in GF(2). They are in Conlon's notation $C_n(T-\gamma)^i$, $C_n(T-\gamma^2)^i$ $i = 0,1,2$. They are projective when restricted to (a), and indecomposable when restricted to P. These six modules are permuted amongst themselves by Ω. (see 2.16). From this it also follows that $C_n(T-0)^N$ is its own projective kernel.

Brauer in Brauer(2) Prop 6 has proved some results about groups whose 2-sylow subgroup is elementary abelian of order 4 : we summarise them.*

(2.22) $B_0(G)$ has four ordinary irreducible characters and three modular. Call the ordinary characters χ_i $i = 0,1,2,3$

Then for all i χ_i is constant $= \pm 1$ on elements of even order and if u is of even order $\chi_i(u) \equiv \chi_i(1)\ (4)$

For elements v of odd order we have the relation

$$1 + \chi_1(v)\,\chi_1(v) + \chi_2(v)\,\chi_2(v) + \chi_3(v)\,\chi_3(v) = 0$$

*Here and throughout such a group will be taken not to be 2-nilpotent.

Chapter 3 Classification of RC_2 modules

Let R be a 2-adic ring as in chapter 2 Suppose $2R = (\pi^e)$, so that e is the index of ramification of 2 in R. π is a generator of the unique maximal ideal of R. Let $(a) = C_2$ be the cyclic group of order 2. Then we have

Theorem 3.1

R(a) has 2+e indecomposable modules. They represent a by

$$1 \qquad -1 \qquad \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \pi \\ 0 & -1 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} 1 & \pi^{e-1} \\ 0 & -1 \end{bmatrix}$$

We shall denote them respectively by

$$R \qquad R^{-} \qquad A(\pi^0) \qquad A(\pi) \quad \cdots \quad A(\pi^{e-1})$$

Proof

That these modules are R(a) modules is routine.

If for example $A(\pi^i)$ were decomposable, there would exist a matrix unimodular over R such that
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & \pi^i \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This is because the composition factors of $A(\pi^i)$ are R and R^-.

This relation forces $c = 0$ and $a\pi^i - b = b$ i.e. $a\pi^i = 2b$

$c = 0$ implies ad a unit in R as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is unimodular and so a itself is a unit (non units form an ideal). Thus $\pi^{i+1} \nmid a\pi^i$

But $i < e$ implies that $\pi^{i+1} \mid 2b$. This contradiction means that no such matrix can exist. i.e. $A(\pi^i)$ is indeed indecomposable.

$A(\pi^0)$ is just the projective indecomposable, R(a) itself.

Now suppose M is any indecomposable representation of a over R. It has a submodule $T(M) = \{ m : ma = m \}$. T(M) is the trivial submodule of M and is R-pure in M. Let m be any element of M. $m(1+a)a = m(a+a^2) = m(a+1)$ and so $m(1+a) \in T(M)$. Thus modulo T(M) a acts by $\overline{m}a = -\overline{m}$

Choose a basis for T(M) and extend it to one for M (this is possible as T(M) is pure in M). Writing operators on the left M(a) has the form

$$M(a) = \begin{array}{c} \\ t \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & N & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right] s$$

N is a t X s R-matrix. R is a principal ideal domain and so there exist unimodular t X t and s X s matrices X and Y such that XNY has the form

$$XNY = \uparrow t \downarrow \left[\begin{array}{ccc} \pi_1^{e_1} & & \\ & \ddots & \\ & & \pi_r^{e_r} \\ & & \end{array}\right] \qquad \pi_i = u_i \pi \;, \; R u_i = R$$

$$\longleftarrow s \longrightarrow$$

(see for example 16.6 of C.R.) So M is R-isomorphic to the representation which takes a to the matrix

$$\left[\begin{array}{c|c} X & 0 \\ \hline 0 & Y^{-1} \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & N & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & 0 & & & \ddots & \\ & & & & & -1 \end{array}\right] \left[\begin{array}{c|c} X^{-1} & 0 \\ \hline 0 & Y \end{array}\right]$$

Multiplying this out we get

$$M'(a) = \left[\begin{array}{ccc|ccc} 1 & & & \pi_1^{e_1} & & \\ & \ddots & & & \ddots & \pi_r^{e_r} \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & -1 & \\ & & & & & \ddots \\ & & & & & -1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & XNY & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & -1 & \\ & & & & & \ddots \\ & & & & & -1 \end{array}\right]$$

Let a basis which affords this representation be $a_1\ a_2\ \ a_t, b_1\ b_2\ \ \ b_s$

Rearranging this basis as follows

$a_1\ b_1\ , a_2\ b_2\ \ , a_r\ b_r, a_{r+1}, a_t, b_{r+1},\ b_s$ a is represented as follows

$$M'(a) = \left[\begin{array}{cccccccc} \begin{matrix} 1 & \pi_1^{e_1} \\ 0 & -1 \end{matrix} & & & & & & & \\ & \ddots & & & & & & \\ & & \begin{matrix} 1 & \pi_r^{e_r} \\ 0 & -1 \end{matrix} & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ & & & & & & & & \ddots \\ & & & & & & & & & -1 \end{array}\right]$$

Since M is indecomposable it must be one of R , R^- or a representation which takes a to $\begin{bmatrix} 1 & \pi^{t} \\ 0 & -1 \end{bmatrix}$ Let the basis affording this be a_1 , b_1 .

Then the basis a_1 , u b_1 , affords the representation $a \longrightarrow \begin{bmatrix} 1 & \pi^{x} \\ 0 & -1 \end{bmatrix}$ and so we need only consider modules of this type.

Suppose $x > e$. i.e. $\pi^x = 2y$ for some y in R. Then the unimodular matrix $\begin{vmatrix} 1 & y \\ 0 & 1 \end{vmatrix}$ transforms the above module as follows

$$\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2y \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus all representations of the form $a \longrightarrow \begin{bmatrix} 1 & \pi^{x} \\ 0 & -1 \end{bmatrix}$ are decomposable if $x > e$. This means that any indecomposable is one of those on our list. It is a simple matter now to show that these are all distinct :

Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is unimodular over R and satisfies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & \pi^{i} \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & \pi^{j} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This forces relations

$c = -c \qquad a\pi^{i} - b = b + \pi^{j} d$

and so $c = 0$ and $a\pi^{i} - \pi^{j} d = 2b$ i.e. $\pi \mid a$ if $j > i$ and $\pi \mid d$ if $j < i$

$c = 0$ implies that ad and so a and d are units. This contradiction completes the proof of 3.1.

Chapter 4 Blocks of defect 1

Let G be a group with 2-sylow subgroup $P \cong C_2 \times C_2$ and suppose that $C_G(P) = P$. We assume that G is not 2-nilpotent, and so $N_G(P) \neq P$. Then $N = N_G(P) \cong A_4$ and if $P = \{1, a, b, c\}$, $C = C_G(a)$ is 2-nilpotent, with complement U say, of order n. The product (a).U is direct and b acts on U without fixed points. Thus U is abelian and b inverts all elements of U. Let $\{ \varphi_i \quad i = 0, \; n-1 \}$ be the linear representations and characters of U in R.

Then RU has idempotents $\quad e_i = e_{\varphi_i} = \frac{1}{n} \sum_{u} \varphi_i(u)^{-1} u$

We can now classify RC modules of vertex (a).

Theorem 4.1 (1)

C has $n + e + (n-1)(e-1)/2$ indecomposable R-representations of vertex (a)

They will be denoted by the following symbols

$\widetilde{R_{(a)}}^P$, $\widetilde{\bar{R}_{(a)}}^P$, A_1 , A_2 $\quad A_{(n-1)/2}$, $\bar{A_1}$, $..\bar{A}_{n-1/2}$,$\widetilde{A(\pi^i)}^P$,$A(\pi^i, j)$

with $i = 1, \ldots e-1$, $j = 1, \ldots n-1/2$

We have the relations

$$R_{(a)}{}^C \cong \widetilde{R_{(a)}}^P \oplus 2(A_1 \oplus \ldots \oplus A_{n-1/2})$$
$$\bar{R}_{(a)}{}^C \cong \widetilde{\bar{R}_{(a)}}^P \oplus 2(\bar{A_1} \oplus \ldots \oplus \bar{A}_{n-1/2})$$
$$A(\pi^i)^C \cong \widetilde{A(\pi^i)}^P \oplus 2(A(\pi^i,1) \oplus A(\pi^i,2) \oplus \ldots \oplus A(\pi^i, n-1/2)) \quad i = 0,1 \,..\, e-1$$

Matrix representations of these modules will be given in the course of the proof. The modules $A(\pi^0, j)$ are projective.

Proof

RC modules of vertex (a) are summands in modules induced up from (a): these have been classified in chapter 3. They are

$R_{(a)}$, $\bar{R}_{(a)}$, $A(\pi^i)$ $i = 0,1 \ldots e-1$

$R_{(a)}{}^C \cong R(C/(a))$ lifted to C modulo (a), $= R((b).U)$ lifted to C.

This latter module is just $RU^{(b).U} = \oplus \, Re_i{}^{(b).U} =$

$= R\langle e_0 \otimes 1, \; e_0 \otimes b \rangle \oplus \cdots \quad \cdots \oplus R\langle e_{n-1} \otimes 1, \; e_{n-1} \otimes b \rangle$

Each of these affords a C representation with respect to the basis $e_i \otimes 1, e_i \otimes b$, as follows.

$$a \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad b \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad u \to \begin{bmatrix} \varphi_i(u) & \\ & \varphi_i(u)^{-1} \end{bmatrix}$$

For i = 0 we call this $\widetilde{R^P_{(a)}}$, because that is what it is and the others are called A_i. Each of A_i is indecomposable since it is the projective indecomposable when restricted to (b). Also if $i > n-1/2$, with respect to the basis $e_i \otimes b$, $e_i \otimes 1$, in that order, A_i has the same form as $A_{i-(n-1)/2}$ and these two representations are equal. It is now clear that the first relation of 4.1(1) holds. All of these modules are (a)-projective and have dimension two and so have vertex (a).

Let t be the linear representation of C which has (b).U in its kernel and maps a to -1.

$R^{-C}_{(a)} \cong (R_{(a)} \otimes t_{(a)})^C \cong R^C_{(a)} \otimes t$

$= (\widetilde{R^P_{(a)}} \oplus 2(A_1 \oplus A_2 \ldots \oplus A_{n-1/2})) \otimes t$

$= \widetilde{R^{-P}_{(a)}} \oplus 2(A^-_1 \oplus A^-_2 \ldots A^-_{n-1/2})$ where A^-_i represents C as follows

$$a \to \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad b \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad u \to \begin{bmatrix} \varphi_i(u) & 0 \\ 0 & \varphi_i(u)^{-1} \end{bmatrix}$$

That these modules are distinct from the first batch is clear by comparing the values of their respective characters at a. Thus we can write the second relation of the theorem.

Lastly, consider $A(\pi^i)^{(a)} \times U = \bigoplus_{\gamma} A(\pi^i) \otimes \varphi_j$ where $\otimes$ here means the external tensor product defined on the direct product of two groups.

Let a basis for each of these summands be u_j, v_j, affording the representation $a \to \begin{vmatrix} 1 & \pi^i \\ 0 & -1 \end{vmatrix}$ $\quad u \to \begin{vmatrix} \varphi_j(u) & \\ & \varphi_j(u) \end{vmatrix}$

Inducing this up to C with respect to the basis $u_j \otimes 1$, $v_j \otimes 1, u_j \otimes b, v_j \otimes b$ it affords

$$a \to \begin{vmatrix} 1 & \pi^i & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \pi^i \\ 0 & 0 & 0 & -1 \end{vmatrix} \quad b \to \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \quad u \to \begin{bmatrix} \varphi_j(u) & & & \\ & \varphi_j(u) & & \\ & & \varphi_j(u)^{-1} & \\ & & & \varphi_j(u)^{-1} \end{bmatrix}.$$

This is again an indecomposable representation which we call $A(\pi^i, j)$

In the case that $j = 0$ it is just the module $\widetilde{A(\pi^i)^P}$. Also rearranging the basis elements in the case that $j > n-1/2$ it becomes obvious that $A(\pi^i, j) \cong A(\pi^i, j-\frac{n-1}{2})$ and so we can write the final relation of 4.1(1) These modules are indecomposable and (a)-projective. Restricted to P they are isomorphic to $A(\pi^i)^P$ and are not projective. It is clear that they have vertex (a).

Theorem 4.1 (2)

C has n-1/2 blocks with defect group (a). We can denote them by b_i and have that A_i, A_i^-, $A(\pi^j, i)$ $j = 0,..e-1$ is a complete set of 2-adic indecomposables in b_i. The module $A(\pi^0, i)$ is the unique projective. A_i, A_i^- afford the only two irreducible characters in b_i and are modularly equal and irreducible. $\overline{A_i}$ is thus the unique modular irreducible in b_i.

Proof

From the representations of A_i, A_i^- it is routine to show that these are modularly equal and irreducible and distinct from any other $\overline{A_j}$. Similarly, $A(\pi^j, i)$ has two copies of $\overline{A_i}$ as its modular constituents; if $j > 0$ in fact it decomposes into $\overline{A_i} \oplus \overline{A_i}$. If we let ς_i, ς_i^- be the (irreducible)character of A_i, A_i^- respectively then the modular decomposition of characters in blocks of C of defect group (a) is given by the following matrix:

$$
\begin{array}{c|c|c|c|c}
 & \overline{A}_1 & \overline{A}_2 & & \overline{A}_{n-1/2} \\
\varsigma_1 & 1 & & & \\
\varsigma_1^- & 1 & & & \\
\hline
\varsigma_2 & & 1 & & \\
\varsigma_2^- & & 1 & & \\
\hline
\vdots & & & \ddots & \\
\hline
\varsigma_{\frac{n-1}{2}} & & & & 1 \\
\varsigma_{\frac{n-1}{2}}^- & & & & 1
\end{array}
$$

Since the decomposition matrix for a block must be "indecomposable" (see C.R. p.609) it follows that ρ_i, ρ_i^- constitute a block of irreducible characters. All the statements of 4.1(2) are consequences of the above remarks.

b_i has only two indecomposable kC modules viz. $\overline{A}_i$ and the projective indecomposable $\overline{A(\pi^o, i)}$. This is proved by considering $k_{(a)}^C$ and $k(a)^C$ and noticing that these have liftings to R that we have already discussed viz. $R_{(a)}^C$ and $R(a)^C$ respectively.

As a corollary to these facts we also have

<u>Theorem 4.2</u>

<u>Any exact sequence</u> $0 \to \overline{A}_i \to E \to \overline{A}_i \to 0$ <u>either splits or else E is projective.</u>

It is now possible to study the blocks of G of defect 1

<u>Theorem 4.3</u>

<u>G has n-1/2 blocks of defect 1 ; each has decomposition matrix</u> $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

<u>Proof</u>

That G has n-1/2 blocks of defect 1 is a consequence of Brauer's first main theorem on blocks and 4.1(2).

Let B be a block of G of defect 1. We take (a) to be its defect group. Any indecomposable kG module in B has vertex (a) or is projective. Consider the Green correspondence between G and C. 2.9 applies and we can say that if M is any kG module of vertex (a) then

$M \in B$ if and only if $f(M) \in f(B) = b_i$ say.

b_i is a block of C of defect group (a) and so has just one non-projective indecomposable namely $\overline{A}_i$. It follows that B has just one indecomposable kG- module with vertex (a). Irreducibles in B cannot be projective since then B would be of defect 0. It follows that B has exactly one modular irreducible and it is $g(\overline{A}_i)$. Also, as any proper quotient of a projective indecomposable is a non projective indecomposable, the only proper quotient of the projective "hull" of $g(\overline{A}_i)$ must be $g(\overline{A}_i)$ itself This means that the unique projective indecomposable in B is an extension of $g(\overline{A}_i)$ by itself. The Cartan invariant is in consequence, 2. The only possibility for the decomposition matrix is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let the two ordinary irreducibles in B be χ_i^+ χ_i^-. They have dimension divisible by 2 by the remarks preceding 2.4 but cannot have $\chi_i^{\pm}(1) \equiv 0(4)$, since this implies that they lie in a block of defect 0. Thus we must have $\chi_i^{\pm}(1) \equiv 2(4)$ Since $\mathfrak{X} = \{1\}$ for the Green correspondence between G and C the correspondent of any RG-module affording χ_i must also have dimension $\equiv 2(4)$. Of all the modules in b_i the only ones satisfying this are A_i, $\overline{A_i}$ and so these must correspond to the only two possible RG modules affording χ_i and χ_i^-.
Also as projective characters vanish on 2-singular elements we must have $\chi_i(x) = 2$ and $\chi_i(x) = -2$, for x 2-singular.

Chapter 5 The Principal Block

Throughout this chapter let G be any group with 2-sylow subgroup $P \cong C_2 \times C_2$,$P = \{1,a,b,c\}$ say.We suppose consistently that G is not 2-nilpotent.It follows that $C_G(P) = P \times O$ and $N(P)/O \cong A_4$. Also $C = C_G(a)$ is 2-nilpotent; $O_{2'}(C) = U$ say .$O \subseteq U$ and $C = P.U.$
We shall determine the decomposition matrix of $B_0(G)$.
Results of Brauer (see 2.22) tell us that B_0 has three modular irreducibles : k_G , F, E.

Theorem 5.1

k_G ,F and E have vertex P.

Proof

The result is clear for k_G.F and E cannot be projective for then B_0 would be a block of defect O and only have one irreducible,which is not the case.So if F does not have vertex P it must have vertex (a). Now consider the Green correspondence between G and C. The kC module f(F) is defined and 2.19 and the remarks following it imply that f(F) must in fact be $\widetilde{k_{(a)}}^{P}$. There is an exact sequence

$$0 \to k_{(a)} \to k(a) \to k_{(a)} \to 0$$

Inducing this up to P and then lifting to C we obtain an exact sequence : $0 \to \widetilde{k_{(a)}}^{P} \to \widetilde{kP} \to \widetilde{k_{(a)}}^{P} \to 0.$

$\widetilde{kP}$ is the principal projective indecomposable of kC. Thus $\widetilde{k_{(a)}}^{P}$ is its own projective kernel and so F also must be its own projective kernel.i.e. some extension of F by itself is projective, and so liftable to R. The character of this module cannot be irreducible since we cannot have a projective irreducible character in a block not of defect O.The only other possibility is that it is a sum of two irreducibles each of which is modularly isomorphic to F. Now as F has vertex (a) its dimension is divisible by 2. However 2.22 forces the dimension of an irreducible character to be odd. It follows that this second possibility also does not arise. Thus F (and similarly E) has vertex P.

An alternative method above would be to use the fact that $\widetilde{k_{(a)}}^{P}$ is liftable to R to show that F would be liftable to R and the same contradiction results.

Seotion 1

For this section it is assumed that one of F, E has odd dimension Suppose F has odd dimension. 2.15 and 2.17 now say that f(F) is one of k_γ k_{γ^2} or $\Omega^{v}(k_{\gamma^i})$. None of these modules is self contragredient. Since $k_\gamma^* = k_{\gamma^2}$ and $*\Omega = \Omega^{-1}*$ this is clear. Thus F is not self contragredient and as F* must also be an irreducible of B_0 it must be the case that E = F*. That is both F and E have odd dimension. Now consider the group (τ, *) of module operators in its action on the modules of G . The orbit of F includes E and must consist of only these two elements. This is because modules in the same orbit as an irreducible must be irreducible, τ and * preserve the principal block and B_0 has only two non trivial irreducibles. These operators commute with the Green correspondence between G and N . In 2.14 there is a diagram which shows the action of (τ , *) on the odd dimensional kN modules. This and the fact that F and E comprise an orbit implies that $f(F), f(E) = k_\gamma, k_{\gamma^2}$ or $\Omega^{v}(k_N)$, $\Omega^{-v}(k_N)$ $v > 0$
In the second case we must have $F, E = \Omega^{\pm v}(k_G)$ and so are liftable to the RG modules $\Omega^{\pm v}(R_G)$. 2.11 shows that the characters of these modules are equal. This cannot be , as they are modularly distinct(= F and E). It follows that the correspondents $f(F)$, $f(E) = k_\gamma$, k_{γ^2} .

Now k_γ , k_{γ^2} are liftable to the RN modules R_γ , R_{γ^2} .We cannot say immediately that this forces F and E to be liftable but this will turn out to be the case. 2.20 implies that $k_{\gamma^i}^G = g(k_{\gamma^i}) \oplus$ Pro $\oplus$ Non where Pro is projective and Non lies totally outside the principal block. Consider $g(R_\gamma)$. $R_\gamma^G = g(R_\gamma) \oplus \mathfrak{X}$ projectives. Induction commutes with the bar map"—" and $\mathfrak{X}$-projective RG modules are modularly $\mathfrak{X}$-projective and so we can write $\overline{g(R_\gamma)} = g(k_\gamma) \oplus$ Pro' $\oplus$ Non' where Pro' ,Non' are summands of Pro and Non. $g(R_\gamma)$ is in B_0 and so Non' = 0. 2.2 implies that Pro' = 0 and so $\overline{g(R_\gamma)} = g(k_\gamma) = F$. Thus we have that F and E are liftable to RG modules with characters χ' and χ'' say. These two characters are modularly distinct and irreducible and so the first three rows of the decomposition matrix are :

	k_G	F	E
χ_0	1	0	0
χ'	0	1	0
χ''	0	0	1

Now consider $\Omega(k_N)$ It has a submodule series $\begin{matrix} k_\gamma \oplus k_{\gamma^2} \\ k_N \end{matrix}$

Inducing this to kG and using 2.10 and 2.20 we get that the induced module has the form

$$\Omega(k_N)^G = \begin{matrix} F \oplus E \\ k_G \end{matrix} \oplus \text{Non} \oplus \text{Pro}$$

None of k_G , F , E can be a summand in the module on the right as none of k_N , k_γ , k_{γ^2} is a summand in $\Omega(k_N)$. It follows that the correspondent $\Omega(k_G)$ of $\Omega(k_N)$ has submodule series $\begin{matrix} F \oplus E \\ k_G \end{matrix}$

and so has composition length 3.

This module is liftable to the RG module $\Omega(R_G)$ with character χ say. $\chi + \chi_0$ is the principal projective character.If the fourth and last row of the decomposition matrix (see 2.22) is $\overline{\chi'''} \approx x k_G \oplus y F \oplus z E$

it follows that the principal projective character is $x\chi''' + \chi_0$

Thus we must have $\chi = x\chi'''$. Since the modular constituents of χ each have multiplicity 1, $x = 1$ and $\chi = \chi'''$.

thus we can complete the decomposition matrix as

$D_0 =$

	k_G	F	E
χ_0	1	0	0
χ'	0	1	0
χ''	0	0	1
χ	1	1	1

Section 2

Now suppose that F and E both have even dimension.

Let $\tau : k \to k$ be the automorphism $x \to x^2$, x in k. The effect of this operator on modules is to preserve irreducibility and the principal block and so $F^{\tau} =$ F or E . In either case we must have $F^{\tau^2} = F$ and $E^{\tau^2} = E$ but suppose $F^{\tau} = F$ and $E^{\tau} = E$. Then 2.3 applies and tells us that F and E are realisable in the fixed field of τ that is, GF(2). Consider the Green correspondence between G and N. As F and E have vertex P their correspondents are defined, and are also realisable in GF(2); also absolutely indecomposable as F and E are absolutely irreducible. 2.21 forces f(F) ,f(E) to be of dimension 6n and their own projective kernels. So F and E are their own projective kernels. It has already been shown in the proof of 5.1 that this cannot be the case. Since $F^{\tau^2} = F$, 2.3 has the force that F (and E) is realisable in the fixed field of τ^2 that is GF(4), but not in GF(2). 2.21 gives a list of the possibilities for f(F) , f(E). They are projective when restricted to (a) - in fact free. Let f(F) = X It will now be shown that X has no trivial composition factors. Suppose not. X has loewy length 2 and so k_N must occur as a factor in either its head or foot. Suppose it occurs in the head of X. Then there is an epi $X \xrightarrow{h} k_N \to 0$. Schur's lemma says that $(F , k_G) = 0$ and 2.5 implies from this that $(X,k_N)/(X ,k_N)_{\mathfrak{X},N} = 0$. Thus our epimorphism h must be $\mathfrak{X}$-projective. That is, there is an (a) map $t : X \to k_N$ such that $T_{(a),N}(t) = h$. Now $X|_{(a)}$ is projective and so any (a)-map t is projective. Then h must be projective also. Now 2.6 says that such a projective map as h cannot be non zero. That is, h is not an epimorphism as assumed. The map h cannot therefore exist. In a similar way X has no trivial submodules. This completes the proof that f(F) and f(E) have no trivial composition factors.

Now consider a projective presentation of X. If the head of X has n summands k_γ, and m k_{γ^2}'s then X is a quotient of $nP_1 \oplus mP_2$.

It is a property of minimal presentations that $s(nP_1 \oplus mP_2)$ belongs to the kernel and so X is a quotient of the direct sum $nP_1/s_1 \oplus mP_2/s_2$

As X has no trivial submodules the submodules U_1, U_2 must also be in the kernel (see 2.14). Thus X is a quotient of the following sum:

$$n\begin{pmatrix} k_\gamma \\ k_{\gamma^2} \end{pmatrix} \oplus m\begin{pmatrix} k_{\gamma^2} \\ k_\gamma \end{pmatrix}$$

The foot of this module has the same dimension as its head = head of X. 2.16 implies that the head and foot of X have the same dimension. It is necessary therefore that X coincides with the module above. Since X is indecomposable m = 1 or n = 1 , n = 0 ,m = 0 respectively. These two modules obtained must now be the correspondents f(F),f(E). They are the two-dimensional modules P_1/U_1, P_2/U_2 of 2.14

These modules are free on subgroups of order 2. Applying Mackey 2.1 to $f(F)^G_N$ it follows that $F_N = f(F) \oplus$ projectives. It will later be shown that this also holds for odd dimensional irreducibles.However in the present case 2.1 also gives that $f(F)^G = F \oplus$ projectives.

An argument in the first section of this chapter has shown that the correspondents $g(k_\gamma)$, $g(k_{\gamma^2})$ are liftable to R. This argument remains valid in the present case and so $\Omega g(k_\gamma) = g(\Omega(k_\gamma))$ and similarly $\Omega g(k_{\gamma^2}) = g(\Omega(k_{\gamma^2}))$ are liftable to R. These will now be shown to have composition length 2.

Consider $\Omega(k_{\gamma^2})$. The results of 2.14 show that this has submodule series $\begin{matrix} k_N \\ \text{—} V_2' \\ 0 \end{matrix}$ The submodule V_2' is easily seen to be the module P_1/U_1 of 2.14 ,that is f(F). We consider the series of submodules of $\Omega(k_{\gamma^2}) \supset V_2' \supset 0$

and induce this to G.

Using the previous results of this section and 2.20 it emerges that
$\Omega(k_{\gamma^2})^G = \begin{smallmatrix} k_G \\ F \end{smallmatrix} \oplus \mathrm{Pro} \oplus \mathrm{Non}$. From this the submodule structure of $g(\Omega(k_{\gamma^2}))$ becomes clear viz. $g(\Omega(k_{\gamma^2})) = \begin{smallmatrix} k_G \\ F \end{smallmatrix}$ and so has composition length 2.

Now the fact that this is liftable to R implies that there is a character χ' with modular decomposition $k_G \oplus F$. Since irreducible characters have odd dimension (2.22) and F has even dimension this character must be irreducible. A similar argument applies to k_γ and E and the characters we obtain are distinct because modularly distinct. We can now fill in the first three rows of the decomposition matrix as follows

	k_G	F	E
χ_0	1	0	0
χ'	1	1	0
χ''	1	0	1

Now consider $\Omega^2(k_N)$. It has foot $k_\gamma \oplus k_{\gamma^2}$ and head $k_N \oplus k_\gamma \oplus k_{\gamma^2}$, as in 2.14. It has a unique submodule M with trivial quotient. M has no trivial composition factors and $s(M) = s(\Omega^2(k_N))$ because it contains the latter. An argument earlier in this chapter now implies that
$M = \begin{smallmatrix} k_\gamma \\ k_{\gamma^2} \end{smallmatrix} \oplus \begin{smallmatrix} k_{\gamma^2} \\ k_\gamma \end{smallmatrix}$. As a consequence $\Omega^2(k_N)$ has a submodule series $-\quad \begin{smallmatrix} k_N \\ f(F)\oplus f(E) \end{smallmatrix}$. We induce this up to G. Using 2.20 and results earlier in this section we can decompose the module obtained as follows : $\begin{smallmatrix} k_G \\ F \oplus E \end{smallmatrix} \oplus \mathrm{Pro} \oplus \mathrm{Non}$

The module $\begin{smallmatrix} k_G \\ F \oplus E \end{smallmatrix}$ must clearly be $g(\Omega^2(k_N)) = \Omega^2(k_G)$ and is liftable to R. $\Omega^2(R_G)$ is such a lifting : let it have character χ

χ has modular constituents k_G, F, E . By inspecting the decomposition of χ_0, χ', χ'' obtained so far no sum of

χ_0, χ', χ'' can possibly have these as modular constituents. The fourth and last row of D_0 must involve k_G since otherwise the fourth irreducible would have even dimension. Also no sum of χ_0, χ', χ'' can have modular constituents $F \oplus E$. From these remarks it is clear that the last row of D_0 is 1,1,1 Thus we can write all of D_0 :

	k_G	F	E
χ_0	1	0	0
χ'	1	1	0
χ''	1	0	1
χ	1	1	1

The congruence (mod 4) of the dimension of these characters is determined : $\chi'(1) = 1+|F| \equiv 3(4) \equiv \chi''(1)$, $\chi(1) \equiv 1(4)$

and Brauer's relation 2.22 becomes $1 + \chi(v) - \chi'(v) - \chi''(v) = 0$

In the case where F and E have odd dimension these congruences follow from theorem 5.3 below and we have $|F| = |E| \equiv 1(4)$

$\chi'(1) = \chi''(1) \equiv 1(4)$, $\chi(1) \equiv 3(4)$

Brauer's relation in this case becomes

$$1 + \chi'(v) + \chi''(v) - \chi(v) = 0$$

We summarise the argument of these sections in

Theorem 5.2

One of the two cases holds :

(1) E and F have odd dimension, $F^* \simeq E \simeq F^{\tau}$, and $D_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(2) E and F have even dimension $F^* = F$, $E^* = E = F^{\tau}$ and $D_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

That $F^{\tau} = E$ in both cases is clear from inspection of the correspondents. Similarly, that $E^* = E$ and $F^* = F$ in the second case.

Theorem 5.3

$F_N = f(F) \oplus$ projectives, and similarly for E.

Proof

When F and E have even dimension this has already been shown above. Assume F and E of odd dimension. We have already shown that in this case $\Omega(k_G)$ has submodule structure $\begin{matrix} & \cdot & \\ F & \oplus & E \\ & \cdot & \\ & k_G & \end{matrix}$. Hence F is a summand in $\dfrac{\phi(P_0')}{\phi^2(P_0')}$

where P_0' is the principal projective indecomposable of G and $\varphi^2(P_0')$ means $\varphi(\varphi(P_0'))$. We regard ϕ as an operator on modules. P_0' must therefore have a quotient with composition series $\begin{matrix} k_q \\ F \end{matrix}$

This quotient module is single headed and so this series is the complete submodule lattice. Call this quotient module X, and consider X_N with series $X' \cdots \begin{matrix} k_N \\ k_\gamma \oplus B \end{matrix} \quad \oplus \text{Pro}$

where B is a sum of modules of vertex (a).

Consider the quotient X'/B. It is an extension $0 \to k_\gamma \to X'/B \to k_N \to 0$

If this extension splits the hypothesis of 2.12 are satisfied with the series $0 \subset B \subset B \oplus k_\gamma \subset X'$

and we conclude that $X' \simeq k_\gamma \oplus \begin{matrix} k_N \\ B \end{matrix}$

X cannot be projective as it has not enough composition factors. It is not self contragredient (by considering the effect of * on the submodule lattice) and this means that it has not vertex (a). It has accordingly vertex P. The equation above would imply by 2.4 that $f(X) = k_\gamma$ which contradicts $f(F) = k_\gamma$ and so the assumption above that the extension X'/B splits is false. That is, X'/B is an indecomposable of dimension 2. Consequently, by 2.14 it is free on (a) and 2.8 implies that X' is a split extension of B by $\begin{matrix} k_N \\ k_\gamma \end{matrix}$

Thus we can write $X_N = \dfrac{P_0'}{V_0} \oplus B \oplus \text{Pro}$.

Now $f(X) = P_0'/V_0$ is free on (a) and so must be X. Thus (see the remarks preceding 2.8) $X_N = f(X) \oplus \text{Pro}$ and the module $B = 0$. Returning now to the origin of B we can conclude that $F_N = k_\gamma \oplus \text{Pro}$ and this completes 5.3.

Chapter 6 Projective indecomposables, Resolutions, and modules of vertex (a)

In this chapter assumptions on G are as in chapter 5. Our aim is to describe kG-modules of vertex (a) in B_0 and the kernels of a projective resolution of k_G .

Lemma 6.1

Let A' be the indecomposable module of vertex (a) in $B_0(G)$

Then $A' \simeq A'^* \simeq A'^{\tau} \simeq \Omega(A')$

Proof

The remarks following 2.19 prove that there is just one such module, and that its correspondent on C is $\widetilde{k_{(a)}}^P$. The lemma clearly holds for the k(a) module $k_{(a)}$. $*$, τ, and Ω all commute with induction of modules and so the lemma holds for ${k_{(a)}}^P$. Likewise these operators all commute with "lifting" modulo U and so 6.1 holds for $\widetilde{k_{(a)}}^P$. Applying the Green correspondence and noticing that these three operators commute with it , we obtain the result stated.

Section 1 F and E of odd dimension

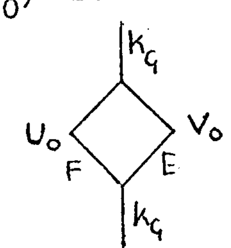
The projective indecomposables of B_0

The last chapter has shown in this case that $D_0 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$

and so the Cartan matrix = $D_0{}^t . D_0 = \begin{matrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{matrix}$

We have already shown in this case that P_0' the principal projective indecomposable has its frattini submodule $\Omega(k_G)$ of the form $F \oplus_{k_G} E$.

Since $s(P_0') \cong P_0'/\phi(P_0')$ it follows that a submodule lattice for P_0' is



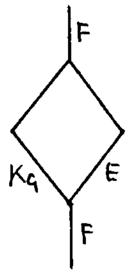
Since F and E are distinct irreducibles it is clear that this must be the complete submodule lattice. The submodule U_0 is single footed and so indecomposable and so $U_0/\phi(U_0) \simeq F$, $\phi(U_0) \simeq k_G$
This means that if P_F is the projective indecomposable having F as quotient, the maximal submodule of P_F has k_G as a quotient. Similarly by considering V_0 there is a uniserial submodule with series

E V_0
k_G

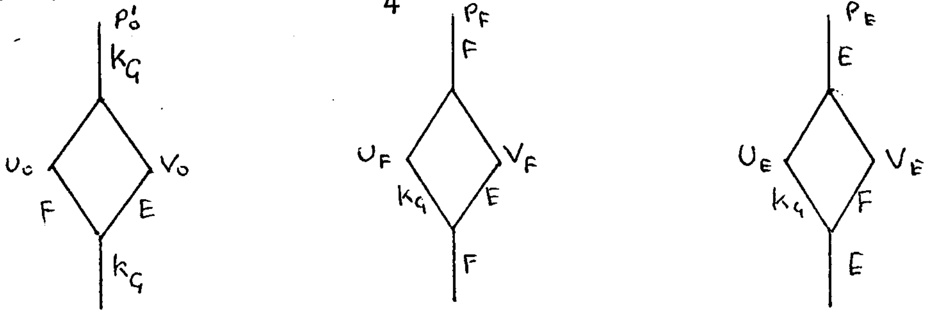
Applying $*$ to this module we obtain a uniserial module with series

k_G
F

This means that P_F which is the injective hull of F has, modulo its socle , a trivial submodule. Let $\phi/s(P_F)$ be the quotient of frattini submodule over socle. Our results now show that this has a quotient and a submodule trivial. Now the matrix C_0 implies that this module has the two composition factors k_G and E and no others. The only way it can have k_G as a quotient and submodule is by being the direct sum of k_G and E. We can now assert that P_F has submodule lattice



A similar result holds for P_E .This submodule lattice is as before unique. We set these modules out to demonstrate the close similarity between them and those for A_4 described in 2.14. :



Inducing the modules of 2.14 to G and using 2.20 would be another way of proving this.

It is clear from these diagrams that the projective modules in B_0 have loewy length 3, and so no module in B_0 has loewy length in excess of 3. In particular projective free modules have loewy length at most 2. (see remarks preceding 2.10)

Theorem 6.2

$\Omega^n(k_G)$ for $n>0$ has loewy length 2. Let $h \in \mathbb{Z}$. Then the head and foot of $\Omega^n(k_G)$ for all n are determined by the following relations:

$\Omega^{-n}(k_G) \cong \Omega^n(k_G)^*$ and

$\Omega^{3h}(k_G)$ has as its loewy factors $\begin{matrix} (h+1)k_G \oplus h(F\oplus E) \\ h\,k_G \oplus h(F\oplus E) \end{matrix}$ A_h

$\Omega^{3h+1}(k_G)$ has as its loewy factors $\begin{matrix} h\,k_G \oplus (h+1)(F\oplus E) \\ (h+1)k_G \oplus h(F\oplus E) \end{matrix}$ B_h

$\Omega^{3h+2}(k_G)$ has as its loewy factors $\begin{matrix} (h+1)\,k_G \oplus (h+1)(F\oplus E) \\ h\,k_G \oplus (h+1)(F\oplus E) \end{matrix}$ C_h

Proof

The proof is by induction on h . $\Omega^{3h}(k_G) = k_G$ is clearly of the form A_h for h = 0. It is necessary to prove that the form A_h implies B_h, B_h implies C_h , and C_h implies A_{h+1} . We prove only this last part, proof of the others being similar.

Accordingly we assume for some $h \geq 0$ the form C_h. A minimal projective presentation of C_h can be lifted from one for its head and so is the direct sum : $Q = (h+1)(P_0' \oplus P_F \oplus P_E)$

$s(Q)$ is contained in the kernel and so the foot of $\Omega^{3(h+1)}(k_G)$ is the head of C_h i.e. $(h+1)(k_G \oplus F \oplus E)$

This is in accordance with the form A_{h+1} . The composition factors of P_0',P_F,P_E are given by the Cartan matrix and so we can evaluate the several multiplicities of the composition factors of Q.

We know the composition factors of C_h by the induction hypothesis and have determined those of $s(\Omega(C_h))$ Those for the head of $\Omega(C_h)$ are now determined by a simple count:

Module	k_G	F	E
C_h	$2h+1$	$2(h+1)$	$2(h+1)$
$s(\Omega(C_h))$	$h+1$	$h+1$	$h+1$
Q	$4(h+1)$	$4(h+1)$	$4(h+1)$
therefore			
$\mathrm{head}(\Omega(C_h))$	$h+2$	$h+1$	$h+1$

It is now clear that the head and foot of $\Omega(C_h)$ are consistent with the form A_{h+1}

<u>The module of vertex (a) in B_0</u>

Consider the unique indecomposable A' of vertex (a) in Bo(N); with loewy factors $\begin{array}{c} k_N \oplus \dot{k}_\gamma \oplus k_{\gamma^2} \\ k_N \oplus k_\gamma \oplus k_{\gamma^2} \end{array}$ and is the module $k_{(a)}{}^{A_4}$ lifted to N.

We induce this series to G and use 2.20 : we get

$$A^G = \begin{array}{c} k_G \oplus \dot{F} \oplus E \\ \cdot \\ k_G \oplus F \oplus E \end{array} \oplus \text{Pro} \oplus \text{Non} = A'' \oplus \text{Pro} \oplus \text{Non}$$

The module A'' is (a) projective. It cannot have any projective summands as it only has loewy length 2. Thus it is a sum of modules of vertex (a) There is in B_0 only one such module and as A'' has only two of each of its composition factors A'' is either indecomposable or a sum of two copies of the same indecomposable which must then have composition factors k_G , F , E . Since F and E have the same dimension this implies that the indecomposable of vertex (a) has odd dimension which contradicts the remarks preceding 2.4. Thus the module A'' must be the indecomposable of vertex (a) in B_0. i.e. A'' = A'.

Section 2 F and E of even dimension

In this case we have that $D_0 = \begin{matrix} 1&0&0 \\ 1&1&0 \\ 1&0&1 \\ 1&1&1 \end{matrix}$ and $C_0 = \begin{matrix} 4&2&2 \\ 2&2&1 \\ 2&1&2 \end{matrix}$

The projective indecomposables of B_0

Let P_0' , P_F ,P_E be the projective "hulls" of k_G , F and E resp. The composition factors of these modules are given by the Cartan matrix. P_F has its head and foot isomorphic to F and so the section $\phi/s(P_F)$ has composition factors 2 k_G , E . Call this module X. X is self-contragredient as F is and so P_F is (see chapter 5). We shall show that X is indecomposable , for suppose not. Since X has only three composition factors it can only decompose in four ways :

$$k_G \oplus k_G \oplus E \;(a) \qquad \begin{matrix} k_G \\ k_G \end{matrix} \oplus E \;(b) \qquad \begin{matrix} k_G \\ E \end{matrix} \oplus k_G \;(c) \qquad \begin{matrix} E \\ k_G \end{matrix} \oplus k_G \;(d)$$

We shall eliminate each of these.

(c) and (d) are not self-contragredient and so cannot occur

(b) would imply that G has an indecomposable with two trivial composition factors. The kernel of such a module has index 2 in G and this forces G to be 2-nilpotent,contrary to hypothesis.So this case does not arise.

(a) implies that P_F has submodule lattice

$$\begin{matrix} F \\ k_G \oplus k_G \oplus E \\ F \\ 0 \end{matrix}$$

and so G has a module with series $\begin{matrix} k_G \oplus k_G \\ F \\ 0 \end{matrix} = Y$

This is obviously not projective and not self-contragredient and so by 6.1 cannot have vertex (a). It has , in consequence, vertex P and its Green correspondent on N is defined. $F_N = \begin{matrix} k_\gamma \\ k_{\gamma^2} \end{matrix} \oplus$ projectives (5.3)

Restricting the module Y and its submodule series above to N we get

$$Y|_N \cong \begin{array}{c} k_N \oplus k_N \\ k_\gamma \\ k_{\gamma^2} \end{array} \oplus \text{Pro} \cong Y' \oplus \text{Pro}$$

Y' cannot have (1)- or (a)-projective summands because its dimension is only 4. It must therefore be $f(Y)$. Its head contains the 2 copies of k_N and its foot contains k_{γ^2}. There are the following two possibilities :

$$\text{foot}(Y') = k_{\gamma^2} \qquad \text{head}(Y') = k_N \oplus k_N \oplus k_\gamma$$

$$\text{foot}(Y') = k_\gamma \oplus k_{\gamma^2} \qquad \text{head}(Y') = k_N \oplus k_N$$

In the first case Y' is single footed of foot k_{γ^2} and so can be embedded in P_{γ^2} . However P_{γ^2} has just one k_N composition factor and so this cannot happen. In the second case Y' has foot $k_\gamma \oplus k_{\gamma^2}$ and so can be embedded in $P_\gamma \oplus P_{\gamma^2}$. The image of Y' must be in the submodule $U_1 \oplus U_2$ as only $k_N \oplus k_N \leq$ in its head. By dimension Y' must actually be $U_1 \oplus U_2$ and so is decomposable. Since our original module Y was indecomposable this also cannot be the case.

The result of this is that a module of the form assumed for Y does not occur. None of the four cases postulated above is the case and so the module X is indecomposable.

$s(X) \leq \phi(X)$ and as X is self-contragredient $s(X)^* \cong X/\phi(X)$. Since $E^* \cong E$ occurs only once as a composition factor of X , E is a factor of neither $s(X)$ nor $X/\phi(X)$. Each of these must therefore be k_G.

X is uniserial with series factors k_G , E, k_G. It now follows from the definition of X that P_F is uniserial with series as follows; a similar argument holds for P_E :

$$P_F \; \begin{array}{|l} F \\ k_G \\ E \\ k_G \\ F \end{array} \qquad\qquad P_E \; \begin{array}{|l} E \\ k_G \\ F \\ k_G \\ E \end{array}$$

We now turn our attention to P_0', the principal projective indecomposable of G. We shall have recourse to the following facts :

$P_0'^* \simeq P_0' \simeq P_0'^{\tau}$. P_0' has composition factors $4k_G$, 2F, 2E, with k_G as its head and foot. $\phi/s(P_0') \simeq \phi/s(P_0')^* \simeq \phi/s(P_0')^{\tau}$ and has composition factors $2k_G$, 2F , 2E . We call this module X. It has no k_G in its head or foot since this would imply the existence of a non-splitting extension of k_G by itself which as before implies that G is 2-nilpotent. So any irreducible summands of X must be F or E. If one occurs, both must, as X is fixed by the module operator τ . Suppose that this is the case viz. $X = F \oplus E \oplus X'$ say. X' has composition factors $2k_G$, F , E, and must be isomorphic to its image under τ as $F \oplus E$ and X both are. Also X' has no trivial factors or submodules since then X would and we have seen that this is not permissible. It is now a consequence of these remarks that X' has series

$$\begin{array}{c} F \\ k_G \oplus k_G \\ E \end{array} \qquad \text{or} \qquad \begin{array}{c} E \\ k_G \oplus k_G \\ F \end{array}$$

Neither of these is compatible with $X'^{\tau} \cong X'$. The result of this argument is that X has no irreducible summands. This has the consequence that $s(X) \subseteq \phi(X)$. Since $X^* \simeq X$, $s(X)^* \simeq X/\phi(X)$. Now if F or E occurs in s(X) both must occur as $X^{\tau} \cong X$. Likewise both must occur in its head. Since F and E occur only twice each as composition factors of X we must have that $s(X) \simeq F \oplus E \cong X/\phi(X)$ and X has the unique loewy series

$$\begin{array}{c} F \oplus E \\ k_G \oplus k_G \\ F \oplus E \end{array}$$

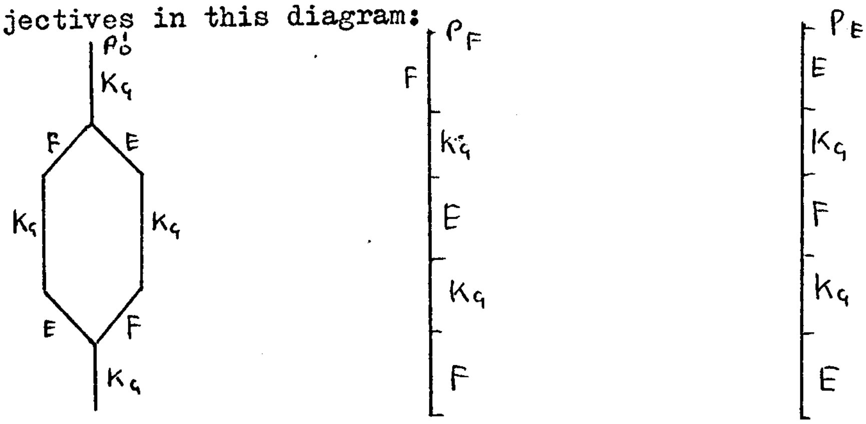
The middle factor here - $\phi/s(X)$ is an extension of k_G by k_G and so must be a trivial extension.

Now because the head of $X = F \oplus E$ we can lift the natural minimal presentation of the latter to X. That is X is a quotient of $P_F \oplus P_E$. $s(P_F \oplus P_E) = F \oplus E$ is in the kernel of this presentation, and so X is a quotient of the module with structure $\begin{array}{|l}F\\K_q\\E\\K_q\end{array} \oplus \begin{array}{|l}E\\K_q\\F\\K_q\end{array}$. X has no trivial submodules and so the foot of this module ($= k_G \oplus k_G$) must also be in the kernel. Thus X is a qotient of the direct sum $\begin{array}{|l}F\\K_q\\E\end{array} \oplus \begin{array}{|l}E\\K_q\\F\end{array}$

This module has the same composition factors as X and so coincides with it. That is, X is the direct sum of the two uniserial modules:

$$X \cong \begin{array}{|l}F\\K_q\\E\end{array} \oplus \begin{array}{|l}E\\K_q\\F\end{array}$$

We can now construct a submodule lattice for P_0'. We summarise the three projectives in this diagram:



<u>The correspondents of k_γ, k_{γ^2}</u>

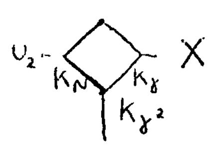
We shall determine $g(k_\gamma)$, $g(k_{\gamma^2})$

<u>Theorem 6.3</u>

$g(k_\gamma)$, $g(k_{\gamma^2})$ <u>are the modules</u> $\begin{array}{|l}E\\K_q\\F\end{array}$, $\begin{array}{|l}F\\K_q\\E\end{array}$

<u>Proof</u>

Consider $\Omega(k_{\gamma^2})$

This has submodule series  where X is an indecomposable of dimension 2. $X \cong f(F)$. Inducing this up to G and using the results of the preceding chapter

we get a module with decomposition $\Omega(k_{\gamma^2})^G = \left| \begin{array}{c} K_q \\ F \end{array} \right.$ $\oplus$Pro$\oplus$Non

Thus $g(\Omega(k_{\gamma^2})) = \left| \begin{array}{c} K_q \\ F \end{array} \right.$

From this $g(k_{\gamma^2}) = \Omega^{-1}\left(\begin{array}{c} \vdots K_q \\ \cdot F \end{array} \right)$ .An injective hull of the module $\left| \begin{array}{c} K_q \\ F \end{array} \right.$ is an injective hull of F as it is clearly single footed.

As P_F is uniserial with series F, K_q, E, K_q, F.

$\Omega^{-1}(g(\Omega(k_{\gamma^2})))$ must be the uniserial module with composition length 3 : $\left| \begin{array}{c} F \\ K_q \\ E \end{array} \right.$

Similarly we have result 6.3 for $g(k_\gamma)$.

<u>A projective resolution of k_G</u>

In chapter 2 the loewy factors of $\Omega^\vee(k_N)$ have been presented.

Let S be the minimal submodule of $\Omega^\vee(k_N)$ such that $\Omega^\vee(k_N)/S$ is trivial, and contragrediently, T the trivial submodule of $\Omega^\vee(k_N)$.

T and S are unique and $0 \subseteq T \subseteq S(\Omega^\vee(k_N)) = \varphi(\Omega^\vee(k_N)) \subseteq S \subseteq \Omega^\vee(k_N)$

Every trivial composition factor of $\Omega^\vee(k_N)$ is accounted for by S or T as $\Omega^\vee(k_N)$ has loewy length 2. S/T has no trivial composition factors

We consider a decomposition of S/T into indecomposables.Inspection of the composition factors of the projective resolution of 2.14 shows that the only indecomposables of odd dimension having no trivial composition factors are k_γ , k_{γ^2}. Here we are also using 2.15. An argument in section (2) of chapter 5 shows that the only even dimensionals with no trivial composition factors are of dimension 2 and are $\begin{array}{c} \vdots k_\gamma \\ \cdot k_{\gamma^2} \end{array}$, $\begin{array}{c} \vdots k_{\gamma^2} \\ \cdot k_\gamma \end{array}$. It follows that we can decompose S/T as

$$S/T \cong f(v)(k_\gamma \oplus k_{\gamma^2}) \oplus g(v)\left(\begin{array}{c} \vdots k_\gamma \\ \cdot k_{\gamma^2} \end{array} \oplus \begin{array}{c} \vdots k_{\gamma^2} \\ \cdot k_\gamma \end{array} \right)$$

The numbers of factors k_γ , k_{γ^2} must be equal as S/T is its own image under the operator Υ . Similarly for the other two factors.

The values of $f(v), g(v)$ will now be shown to determine the composition factors of the projective kernels of G : $\Omega^v(k_G)$. The number of factors of $\Omega^v(k_N)$ isomorphic to k_γ is the sum $f(v) + 2g(v)$ and this has been determined in 2.14. So evaluation of only one of f , g is necessary. This however will not be carried out here. $\Omega^v(k_N)$ has the submodule series

$$\begin{array}{ll} & s\,k_N \;\text{---}\; S \\ & f_v(k_\gamma \oplus k_{\gamma^2}) \oplus g_v\left(\begin{smallmatrix} k_\gamma \\ k_{\gamma^2}\end{smallmatrix} \oplus \begin{smallmatrix} k_{\gamma^2} \\ k_\gamma \end{smallmatrix}\right) \\ T \;\text{---} & t\,k_N \end{array}$$

s and t are determined in 2.14. We induce this up to G and use 2.20 and/or theorem 5.3. The correspondents $g(k_{\gamma^i})$ are shown in 6.3 and $g\left(\begin{smallmatrix} k_\gamma \\ k_{\gamma^2}\end{smallmatrix}\right)$ etc. are the irreducibles in B_0 in this case. Using this and 2.10 we can write a decomposition for $\Omega^v(k_N)^G$ as follows

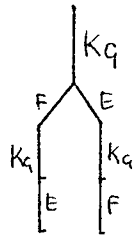
$$\Omega^v(k_N)^G = \begin{array}{ll} & \overline{}\; X \\ & s\,k_G \\ & \overline{}\; Y \\ & f_v\left(\begin{smallmatrix} F \\ k_G \\ E \end{smallmatrix} \oplus \begin{smallmatrix} E \\ k_G \\ F \end{smallmatrix}\right) \oplus g_v(F \oplus E) \\ & \overline{}\; Z \\ & t\,k_G \end{array} \oplus \text{Pro} \oplus \text{Non} \;=\; X \oplus \text{Pro} \oplus \text{Non}$$

$\Omega^v(k_G)$ must now be a summand in X with by 2.20 a projective complement. If P_F or P_E were a summand it would have to be contained in Y as X/Y is trivial. However Y has loewy length not exceeding 4 and as P_F and P_E have loewy length 5 this cannot be. Thus any summand in X must be P_0' . This will also be shown to be impossible. Let ρ be a submodule of X isomorphic to P_0' . If $\rho \cap Z = 0$ then X/Z would have a submodule isomorphic to P_0' which contradicts its loewy length. Therefore, $\rho \cap Z = U = s(\rho) \cong k_G$. By assumption, $X = X' \oplus \rho$ and $U \subseteq \rho$ and so X/U decomposes as $X'' \oplus \rho/U$. Clearly X/U has a series

$$\begin{array}{l} \overline{}\; X' \\ s\,k_G \\ \overline{}\; Y' \\ f_v\left(\begin{smallmatrix} F \\ k_G \\ E \end{smallmatrix} \oplus \begin{smallmatrix} E \\ k_G \\ F \end{smallmatrix}\right) \oplus g_v(F \oplus E) \\ \overline{}\; Z' \\ (t-1)\,k_G \end{array}$$

and P' = P/U is summand

The submodule lattice of P' is



We have that $X' \cong P' \oplus X''$. The submodule Z' of X' must lie in X'' as its projection onto P' lies in $s(P') \simeq F \oplus E$ and so is zero. And so we can write $X'/Z' =$

$$\begin{array}{c} \bullet \text{---- } X' \\ s\,k_G \\ \bullet \text{---- } Y' \\ f_v\begin{pmatrix} F & & E \\ k_G & \oplus & k_G \\ E & & F \end{pmatrix} \oplus g_v(F \oplus E) \\ \bullet \end{array}$$

and has a summand P'

Let $\phi(P') = P''$ and $X'/Z' \simeq P' \oplus X''/Z'$ say $X'/Z' \simeq P' \oplus Q$

Since X'/Y' is trivial and Y'/Z' has no trivial quotients it must be the case that $P'' \subseteq Y'$, and $Q \oplus P''$ contains Y'. Clearly $X'/(Q \oplus P'') \simeq k_G$ and so $Q \oplus P''$ has series.

$$\begin{array}{c} \bullet \\ (s-1)k_G \\ \bullet \text{----- } Y'' \\ f_v\begin{pmatrix} F & & E \\ k_G & \oplus & k_G \\ E & & F \end{pmatrix} \oplus g_v(F \oplus E) \\ \bullet \end{array}$$

Since P'' has no trivial quotients it must lie inside Y''. That is P'' $\overline{(s-1)k_G}$ is asummand in Y'' and now by Krull-Schmidt $Q \cong (Q \oplus P'')/P'' \cong (f_v-1)\begin{pmatrix} F & & E \\ k_G & \oplus & k_G \\ E & & F \end{pmatrix} \oplus g_v(F \oplus E)$

Returning now to the module X with summand P and complement X' and piecing together these facts it is the case that X' must have a series:

$$\begin{array}{c} \bullet\, X' \\ (s-1)\,k_G \\ \bullet \\ (f_v-1)\begin{pmatrix} F & & E \\ k_G & \oplus & k_G \\ E & & F \end{pmatrix} \\ \bullet \\ (t-1)\,k_G \\ \bullet \end{array}$$

By our assumptions $\Omega^{v}(k_G)$ is a summand in this module. Restrict it to N. Using 5.3 and 6.3 it is not difficult that the following holds:

$$\left.\begin{pmatrix} F \\ k_G \\ E \end{pmatrix}\right|_N \simeq k_{\gamma^2} \oplus \text{Pro}$$

A similar relation holds for $g(k_\gamma)$. Applying this to X'_N and using again 2.10 we can decompose it as

$$X'|_N \simeq \begin{array}{c} (s-1)\,k_N \\ (f_v-1)(k_\gamma \oplus k_{\gamma^2}) \oplus g_v\left(\begin{array}{c} k_\gamma \\ k_{\gamma^2}\end{array} \oplus \begin{array}{c} k_{\gamma^2} \\ k_\gamma\end{array}\right) \\ (t-1)\,k_N \end{array} \oplus \text{Pro} \quad = V \oplus \text{Pro say}$$

Since we assumed that $\Omega^v(k_G)$ was a summand in X' we must have that $\Omega^v(k_N)$ is a summand in the module V. A comparison of dimensions however shows that $\dim(\Omega^v(k_N)) = 2v+1 \;\; = s+t+2f_v+4g_v$

$$= (s-1)+(t-1)+2(f_v-1)+4g_v+4 > s-1+t-1+2(f_v-1)+4g_v = \dim(V)$$

and so this cannot be the case.

It follows that our original assumption that P_0' was a summand in X is false and so X is indecomposable, This means that $X = \Omega^v(k_G)$.

By inspecting the submodule series for X the composition factors are clearly as follows :

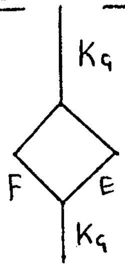
	$s+t$	factors	k_G
	$2f_v+g_v$	factors	F
	$2f_v+g_v$	factors	E

<u>Module of vertex (a) in B_0</u>

As we have already shown, B_0 has exactly one indecomposable of vertex (a)

<u>Theorem 6.4</u>

<u>The unique indecomposable A' of vertex (a) in B_0 has submodule lattice</u>



<u>Proof</u>

Consider the unique kN indecomposable of vertex (a) in $B_0(N)$.It is called A. The remarks following 2.16 show that A has a series

$$\begin{array}{ccc} & A & \\ & K_N & \\ \begin{array}{l} K_\gamma \\ K_{\gamma^2} \end{array} & \oplus & \begin{array}{l} K_{\gamma^2} \\ K_\gamma \end{array} \\ & K_N & \\ & 0 & \end{array}$$

If we induce this up to G we get a decomposition

$$A^G \simeq \begin{array}{c} K_G \\ F \oplus E \\ K_G \end{array} \oplus \text{Pro} \oplus \text{Non} = A'' \oplus \text{Pro} \oplus \text{Non}$$

Here again we are using 2.10 and 2.20. The module A' must be a summand in A" and in fact must be all of it as no projective (loewy length 5) can be summand , and it must be (a)-projective. The submodule lattice above must be complete as if not, A' would have say F in its head and then by lemma 6.1 would have to have F $\oplus$ E in both its head and foot , which are distinct because A' is indecomposable (but not irreducible).

Chapter 7 The case $G = PSL(2,q)$

In this chapter some of the results of the preceding pages will be verified for the case where G is the group $PSL(2,q)$ $q = 3,5 \pmod 8$ The 2-sylow subgroup is $P \cong C_2 X\, C_2$ and is its own centraliser in G Using the character table for G it is routine to distribute the ordinary irreducibles into 2-blocks. Thus B_0 has four and the blocks of defect 1 have two each. These two are modularly irreducible and equal and so the decomposition matrix for such a block is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in agreement with chapter 4.

G has a doubly transitive permutation .Over an ordinary field (for 2-adic ring) this affords a character $\chi_0 + \chi$ where χ_0 is the trivial character and χ is irreducible. Over a field of characteristic 2 it will be shown below that this permutation representation affords a module with composition factors: $2k_G$, F , E where F ane E are irreducible. Also this module is indecomposable and so χ , E and F all belong to the principal block. It follows from this that χ has modular constituents k_G , F and E. Let χ' and χ'' be the other two irreducible characters in B_0

$q \equiv 3 \pmod 8$ $\quad \chi' + \chi'' = \chi - \chi_0$ on 2-regular elements

$q \equiv 5 \pmod 8$ $\quad \chi' + \chi'' = \chi + \chi_0$ on 2-regular elements

In the first case it is necessary that $D_0 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$

In the second case D_0 must be $\begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$

Let k be a field of characteristic 2, F_q the Galois field of order $q = p^n$ with p odd. Let $\Omega = PG(1,q)$ the projective line over F_q, with elements $\{ [1, y], y \in F_q, [0, 1] \}$

$G = PSL(2,q)$ has elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

G acts on Ω by the rule

$$[x, y] \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [xa + yc, xb + yd]$$

Let $M = k\Omega$ be the kG module afforded by Ω.

Theorem 7.1 (Green)

M is indecomposable. $s(M) \cong M/\phi(M) \cong k_G$ and $\phi(M)/s(M) \cong F \oplus E$ where F and E are the two non trivial irreducibles in $B_0(G)$.

Proof

As G is 2-transitive on Ω, End = End(M) has dimension 2 with basis

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} \doteq J,$$

these matrices being of order q+1 X q+1

q+1 is even and k has characteristic 2 and so $J^2 = 0$ and

$kJ = J(\text{End})$ the Jacobson radical of End. $\text{End}/J(\text{End}) \cong k_G$

and so M is indecomposable.

Let $W = \{ \sum k_\omega \omega \mid \sum k_\omega = 0 \}$

and $T = k\Sigma = k \sum \omega$

W and T are submodules of M. $M/W \cong k_G \cong T$ and $T \leq W$ as k has characteristic 2. It will be shown that W has only three non degenerate submodules.

W has basis $\{ [1, y] - [0, 1] = t_y, y \in F_q \}$

Consider $G_b = G_{[0,1]} = \left\{ \begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \right\} \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \right\} = H.S$

Thus S is regular on $\{t_y : y \in F_q\}$. $|S| = q = p^n$

Extend k if necessary to contain a p^{th} root of 1 . k is then a splitting field for S of characteristic prime to $|S|$

Let $\hat{S}$ be the set of all linear representations of S in k. $|\hat{S}| = |S|$

The mapping $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \longleftrightarrow y$ is an isomorphism between F_q^+ and S

If $\theta \in \hat{S}$ we shall write $\theta(y)$ to mean $\theta\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$

Then W has basis $\left\{ \frac{1}{q} \sum \theta(y)^{-1} t_y \,\middle|\, \theta \in \hat{S} \right\} = \{ e_\theta \mid \theta \in \hat{S} \}$

Each element of this basis spans an S submodule of W and any S submodule is a sum of some of these. This is because they are all distinct irreducible kS modules.

If $\theta_0 = 1$, $k e_{\theta_0} = T$ is a G_b (G in fact) submodule of W

Any non-trivial submodule of W is an S submodule and so must contain some $k e_\theta$, $\theta \neq \theta_0$

Hence it also contains $e_\theta \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} = \frac{1}{q} \sum \theta(y)^{-1} t_{yx^{-2}} = \sum \theta(x^2 y)^{-1} t_y$

This spans another S submodule of W viz. that afforded by the linear representation $y \rightarrow \theta(x^2 y)$

In this way H acts on the $\{e_\theta\}$

Let u be a non square in F_q for example a multiplicative generator of F_q^*

The orbits of H on the $\{e_\theta\}$ are $\Theta = \{e_\theta, \cdots e_{\theta'}\}$, $\Psi = \{e_\psi, \cdots e_{\psi'}\}$

Θ and Ψ have equal length q-1/2 .This is because $\theta(x^{-2}y) = \theta(y) \; \forall y \Rightarrow x^2 = 1 \Rightarrow \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So H acts regularly on the S modules $k e_\sigma$ in Θ and Ψ

Suppose $\theta(uy) = \theta(a^2 y)$ then $a^2 = u$ X

It follows that any $\psi \in \Psi$ can be taken as $\psi(y) = \theta(u a^2 y)$

It is now clear that W decomposes over G_b as $k e_{\theta_0} \oplus k\Theta \oplus k\Psi$ and each of these is irreducible.They are distinct as they are when restricted to S. Any kG submodule of W must be a sum of some of these.

$k e_{\theta_0}$ is a submodule = T in fact.

We show that each of $k e_{\theta_0} \oplus k\Theta$, $k e_{\theta_0} \oplus k \Psi$

is a submodule and simultaneously that $k\Theta$, $k\Psi$

are not kG submodules. .

$z = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \notin G_b$. As G is 2-transitive $G = G_b.1.G_b \cup G_b.z.G_b$

Consequently, it is enough to show that $k e_{\theta_0} \oplus k\Theta$, $k e_{\theta_0} \oplus k\Psi$

are closed under the action of z and $k\Theta$, $k\Psi$ are not.

Let $\theta \in \Theta$

$$e_\theta z = \frac{1}{q}\sum \theta(y)^{-1} t_y z = \frac{1}{q}\sum_{y \neq 0} \theta(y)^{-1}(t_{-\frac{1}{y}} - t_0) - t_0$$

$$= \frac{1}{q}\sum_{y\neq 0} \theta(\tfrac{1}{y})\, t_y - \frac{1}{q}\sum \theta(y)^{-1} t_0 = \frac{1}{q}\sum_{y\neq 0}\theta(\tfrac{1}{y})t_y = \frac{1}{q}\sum f(y)t_y$$

where $f(y) = \theta(\frac{1}{y})$, $y \neq 0$; $f(0) = 0$

Now any such function f can be written as $f = \sum_{\sigma \in \hat{S}} (f,\sigma)\,\sigma$

where $(f , \sigma) = \frac{1}{q}\sum f(-y)\,\sigma(y)$

This is because classical character theory works for S and k as

char(k) is prime to $|S|$ and is a splitting field for S.

Applying this to the expression above we get

$$e_\theta z = \frac{1}{q}\sum_{\sigma}\sum_{y}(f,\sigma^{-1})\,\sigma^{-1}(y)\,t_y = \sum_{\sigma}(f,\sigma^{-1})\,e_\sigma$$

We now prove that if $\sigma \in \Psi'$, $(f , \sigma^{-1}) = 0$

and this will imply the final result. σ may be taken as

$\sigma(y) = \theta(\upsilon' y)$, $\upsilon' = \upsilon a^2$ and

thus $(f, \sigma^{-1}) = \frac{1}{q}\sum f(-y)\,\sigma^{-1}(y) = \frac{1}{q}\sum_{y\neq 0}\theta(-\frac{1}{y})\,\theta(-\upsilon' y)$

$$= \frac{1}{q}\sum_{y \neq 0}\theta(\tfrac{1}{y} + \upsilon' y)$$

Each term in this summation occurs twice. To see this, consider the

map $y \rightarrow \frac{1}{y} + \upsilon' y$ of F_q to itself. The inverse image of

any x in F_q is the set of solutions of the quadratic

$$\upsilon' y^2 - x y + 1 = 0$$

This has no repeated roots for if it did we would have

$x^2 - 4\upsilon' = 0$ $\qquad (x/2)^2 = \upsilon' = \upsilon a^2$

This contradicts choice of u as a non-square. Thus each element of F_q either does not occur as an argument in $\theta\left(\frac{1}{y} + u'y\right)$ or else occurs exactly twice. The summation above must therefore be 0

This proves that $ke_{\theta_0} \oplus K\Theta$, $Ke_{\theta_0} \oplus K\Psi$

are indeed kG submodules of W . Also $K\Theta, K\Psi$ cannot be submodules as

$$(f , \theta_0^{-1}) = \frac{1}{q} \sum_{y \neq 0} \theta(Y/y) = \frac{1}{q} \sum_{y \neq 0} \theta(y) = 1$$

That is $e_\theta Z$ involves e_{θ_0}

We have now that W has only three non-degenerate submodules viz.

$$T = Ke_{\theta_0} , \quad T \oplus K\Theta , \quad T \oplus K\Psi$$

As M is a permutation module $M^* \cong M$

It is clear that any trivial submodule of M must be T and

contragrediently, the only submodule of M with trivial factor is W.

From these follows that $W^* \cong M/T$ and so modulo T, M has just

three non-degenerate submodules which must be W/T, $\frac{T \oplus K\Theta}{T}$, $\frac{T \oplus K\Psi}{T}$

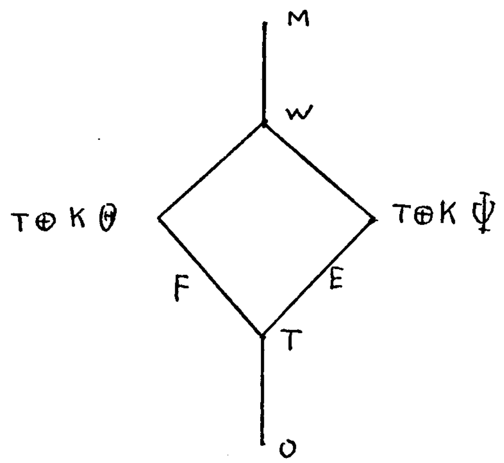
Now let X be any submodule of M.

$$X \cap W = 0 \Rightarrow X \simeq \frac{X \oplus W}{W} \leq \frac{M}{W} \simeq K_q \Rightarrow X \leq T \Rightarrow X = 0$$

On the other hand $X \neq 0 \Rightarrow X \cap W \neq 0 \Rightarrow X \supseteq T \Rightarrow \frac{X}{T} \leq \frac{M}{T} \simeq W^*$

$$\Rightarrow X = T \oplus K\Theta , \ T \oplus K\Psi , \ W , \ M$$

From this it follows that the complete submodule lattice for M is



Now let P be a 2-sylow subgroup of G. $N = N_G(P) \cong A_4$. We shall determine the structure of F_N and E_N

Firstly the case $q \equiv 3\ (8)$

Theorem 7.2

$q \equiv 3(8)$ $|G| = q(q-1)q+1)/2$

$|G_b| = q(q-1)/2$ $M = K\Omega \cong k_{G_b}^G$ and so is projective. It is the principal projective indecomposable of G.

$F_N \simeq k_\gamma \oplus \text{Pro}$ and $E_N \cong k_{\gamma^2} \oplus \text{Pro}$

Proof

G_b has odd order so $P \cap G_b = 1$. Let $P = 1, a, b, c$. P acts regularly on Ω

Let its orbits be $a_1a_2a_3a_4$ $b_1b_2b_3b_4$ b_5b_6 .... b_{q-3}

where we choose the numbering so that a is represented on the indices by $(12)(34),(56)(78)$ etc and b by $(13)(24),(57)(68)$ etc.

Let W' be the three dimensional submodule of W_p spanned by

$a_1-a_2, a_1-a_3, a_1-a_4$ and let Q be spanned by

b_1-a_1	b_5-a_1	$b_{q-6}-a_1$
b_2-a_2	b_6-a_2	$b_{q-5}-a_2$
b_3-a_3	b_7-a_3	$b_{q-4}-a_3$
b_4-a_4	b_8-a_4	$b_{q-3}-a_4$

On each group of four P acts regularly and so the k-space Q spanned by these is a projective submodule of W_p, of dimension $q-3$.

Its sum with W' is clearly direct and so we may write

$W_p = W' \oplus Q$. W' is the module $\Omega(k_p)$. Let T' be its unique irreducible submodule.

$T \cap Q$ is obviously 0 and so $\frac{T+Q}{T} \cong Q$ is a projective submodule of W/T_p and so is a summand. Also $T + Q = T' \oplus Q$ and so

Now $W'/T' \cong \phi/s(kP)$ and so $\cong k_p \oplus k_p$. $\qquad \frac{W}{T+Q} = \frac{W'\oplus Q}{T'\oplus Q} = \frac{W'}{T'}$

That is we have $W/T_p \cong k_p \oplus k_p \oplus \text{Pro}$

7.1 shows that $W/T \cong F \oplus E$ and so $F_p \simeq k_p \oplus \text{Pro}$

and $E_P \cong k_P \oplus \text{Pro}$

Since k_P has vertex P so do F and E .Also $f(F)$, $f(E)$ are summands in $k_P{}^N \cong k_N \oplus k_\gamma \oplus k_{\gamma^2}$.

It is now clear that $f(F), f(E)$ are k_γ, k_{γ^2} .The complement of $f(F)$ in F_P is Pro when restricted to P and so is projective as kN module.

All parts of theorem 7.2 have now been proved.

This verifies the results of chapter 5 (1) and 5.3

Consider $F \otimes F|_N \cong F_N \otimes F_N \cong (k_\gamma \oplus \text{Pro}) \otimes (k_\gamma \oplus \text{Pro})$

$$= k_\gamma \otimes k_\gamma \oplus \text{ projectives } = k_{\gamma^2} \oplus \text{ projectives}$$

$$= f(E) \oplus \text{ projectives}$$

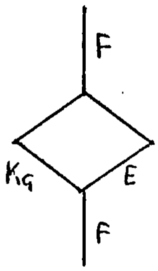
It follows that $F \otimes F \cong E \oplus$ projectives

$F \otimes E = k_G \oplus$ projectives similarly

and $E \otimes E = F \oplus$ projectives

The module M above is the principal projective indecomposable of G. Tensor it with F. This is an exact operation and so we can decompose the product, using 2.10 as

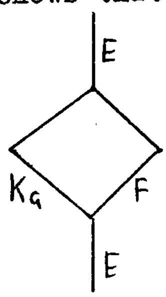
$M \otimes F \cong$



$\oplus$ Pro $= P_F \oplus$Pro

The module P_F is projective,has quotient F and obviously cannot decompose. It must be the unique projective indecomposable having head F and the submodule lattice above is clearly complete.

A similar argument shows that P_E has submodule lattice



This verifies the properties of these lattices deduced in chapter 6 (1)

Now suppose $q \equiv 5 \pmod 8$

This time $|G_b| = q(q-1)/2 \cong 2 \pmod 4$

<u>Theorem 7.3</u>

$M \cong k_{G_b}^G$ <u>is the unique indecomposable in B_0 of vertex</u> (a)

$F_N = C_1(T-\gamma)^0 \oplus \text{Pro}$, $E_N = C_1(T-\gamma^2)^0 \oplus \text{Pro}$ where

$C_1(T-\gamma)^0$ <u>represents N by</u>

$$g \to \begin{bmatrix} \gamma^2 & 0 \\ 0 & \gamma \end{bmatrix} \qquad a \to \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$$

$C_1(T-\gamma^2)^0$ <u>represents N by</u>

$$g \to \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^2 \end{bmatrix} \qquad a \to \begin{bmatrix} 1 & \gamma^2 \\ 0 & 1 \end{bmatrix}$$

<u>Proof</u>

Choose a 2-sylow subgroup P such that $P \cap G_b = \langle a \rangle$ where $P = \{1, a, b, c\}$

As $|G_b| \equiv 2 \pmod 4$ no elements of Ω are fixed by P. As G is 3-regular on Ω a fixes exactly two elements, say A and A'. Similarly let b fix B, B' and c fix C, C'. These must comprise 6 distinct elements of Ω , and clearly must form an orbit of N. Let the other P-orbits be

$b_1\ b_2\ b_3\ b_4\quad a_1\ a_2\ a_3\ a_4\ a_5\ \ldots\ a_{q-9}$ if these exist. Again, we arrange that a is represented on the indices by (12)(34), (56)(78) etc. and b by (13)(24) , (57)(68) etc.Consider the following set of elements

$A-b_1$	a_1-b_1	a_5-b_1	$a_{q-12}-b_1$
$A-b_2$	a_2-b_2	a_6-b_2	$a_{q-11}-b_2$
$A'-b_3$	a_3-b_3	a_7-b_3	$a_{q-10}-b_3$
$A'-b_4$	a_4-b_4	a_8-b_4	$a_{q-9}-b_4$

Each set of four is permuted regularly by P and so spans a projective submodule of W_P . Let W' be the submodule of W_P of elements

$\sum k_\omega \omega \; : \; \omega = A, A', B$ etc. in which $\sum k_\omega = 0$

W' has dimension 5 . The direct sum of the projective modules above - call it Q has dimension q-5 and its sum with W' is direct. Thus $W_P = W' \oplus Q$. Now as before let T be the trivial submodule of W and

T' that of W'. $T + Q = T' \oplus Q$ and $(T+Q)/T \simeq Q$ is projective as $Q \cap T = 0$. Thus W/T_p has projective submodule $Q + T/T$ and its complement is $W/(Q+T) = \dfrac{W' \oplus Q}{T + Q} = \dfrac{W' \oplus Q}{T \oplus Q} \simeq \dfrac{W'}{T'}$

$W/T \simeq F \oplus E$ and so $(F \oplus E)_p \simeq W'/T' \oplus \text{Pro}$

The kN module afforded by A,A',B,B',C,C' is just the module $k^N_{(a)}$ and in the notation of the remarks after 2.16 W'/T' is the section M/I. A decomposition of this module is given in those remarks.

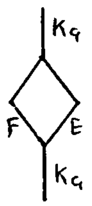
W' and T' are kN modules and using an argument like that in the previous case we can write $F \oplus E|_N = W'/T' \oplus \text{Pro}$

The correspondents f(F), f(E) are now clearly those two 2-dimensional kN modules having no trivial composition factors obtained in the remarks referred to. The notation for these used in the statement of the theorem is as in Conlon(5) This completes 7.3 .

The results of chapter 5 (1) and 5.3 and theorem 6.4 have now been verified. We continue with this case and examine the structure of projective indecomposables.

We have so far that (1) $D_0 = \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$

(2) $M = k^G_{G_b}$ - the indecomposable of vertex (a) has submodule lattice



(3) G_b has order $q(q-1)/2 \equiv 2 \pmod 4$

G_b has a normal 2-complement O say.

$k_O{}^{G_b} = k(G_b/O)$ is an extension of k_{G_b} by itself

Thus k_O^G is an extension of M by M. It is projective and has M and so k_G as a quotient. Thus it involves P_0' the principal projective. Its composition factors are $4k_G$, 2F, 2E and (1) above implies that (4,2,2) is the first column of the Cartan matrix. i.e. the module k_0^G is the principal projective indecomposable of G. Its submodule lattice will now be deduced.

We use the notation of theorem 7.1. Let x be the element $\begin{bmatrix} \upsilon & 0 \\ 0 & \upsilon^{-1} \end{bmatrix}$ of G. We shall use the abbreviation .y for the coset O.Y of O in G. Thus G_b is a union of the cosets .1 , .x and we can list the other cosets as $.z\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$ and $.xz\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$, for $y \in F_q$. S acts

transitively on these latter two sets and its action is obvious. Also S fixes the first two cosets as x normalises $O:(O,x) = G_b$

The action of x is a consequence of the two facts :

$$z\,x = x^{-1} z\ , \quad \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} x = x \begin{bmatrix} 1 & y\upsilon^{-2} \\ 0 & 1 \end{bmatrix}$$

The action of z is determined by the following formulae which straitforward matrix algebra establishes :

$y \neq 0$, y a square in F_q $\quad .z\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} z = .z\begin{bmatrix} 1 & -\frac{1}{y} \\ 0 & 1 \end{bmatrix}$

y a non square in F_q $\quad .z\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} z = .xz\begin{bmatrix} 1 & -\frac{1}{y} \\ 0 & 1 \end{bmatrix}$

$y = 0$ $\quad z^2 = 1$

Notice that as we are assuming that $q \equiv 5 \pmod 8$ -1 is a square. As in the proof of 7.1 we can choose a basis for P_0' consisting of coset sums as follows :

$$.1,\ .x,\ .ze_0,\ .xze_0\ ,\ \{.ze_\theta\}\ ,\ \{.xze_\theta\}$$
$$\{.ze_\psi\}\ ,\ \{.xze_\psi\}$$

The sets $\{.ze_\theta, .xze_\theta x, .ze_\theta x^2 \cdots\} = \Theta$ afford the same irreducible $\Theta' = \{.xze_\theta, .ze_\theta x, .xze_\theta x^2 \cdots\}$ F of G_b

the sets $\{.ze_\psi, .xze_\psi x, .ze_\psi x^2 \cdots\} = \Psi$ afford another irreducible $\Psi' = \{.xze_\psi, .ze_\psi x, .xze_\psi x^2 \cdots\}$ E of G_b

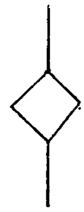
The decomposition of $P_0'|_{G_b}$ is therefore

$$K\langle .1, .x\rangle \oplus K\langle .ze_0, .xze_0\rangle \oplus K\Theta \oplus K\Theta' \oplus K\Psi \oplus K\Psi \cong 2\left(K_0^{G_b} \oplus F' \oplus E'\right)$$

and so P_0' has very limited submodule possibilities. Consider the submodule M' spanned by

$$\cdot 1 + \cdot x\ ,\quad \cdot z e_0 + \cdot xz e_0\ ,\ \{(\cdot z + \cdot xz) e_\theta\}\ ,\ \{(\cdot z + \cdot xz) e_\psi\}$$

This is isomorphic to M and so has submodule lattice

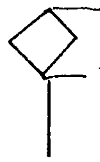


where its socle is spanned by $\cdot 1 + \cdot x + \cdot z e_0 + \cdot xz e_0 = T$

and another two non-degenerate submodules are

$$K\langle (\cdot z + \cdot xz) e_\theta \cdots \rangle \oplus T\ ,\quad K\langle (\cdot z + \cdot xz) e_\psi \cdots \rangle \oplus T$$

Let the sum of these be Q. Then Q has submodule lattice



We shall show that $Q = s^2(P_0')$ and that $s(P_0'/Q) \simeq k_G \oplus k_G$

Consider P_0'/Q . Suppose it had an irreducible submodule $L/Q \not\simeq k_G$

$L/Q \simeq F$ say , and $L|_{G_b}$ must be of the form

$$L|_{G_b} = T \oplus K\langle\{(\cdot z + \cdot xz) e_\theta\}\rangle \oplus K\langle\{(\cdot z + \cdot xz) e_\psi\}\rangle \oplus F'$$

where F' is another submodule of $K\langle\{\cdot z e_\theta\},\{\cdot xz e_\theta\}\rangle$

and so $F' \oplus K\langle\{(\cdot z + \cdot xz) e_\theta\}\rangle = K\langle\{\cdot z e_\theta\},\{\cdot xz e_\theta\}\rangle$

since the latter has composition length 2. Thus $\cdot z e_\theta \in L$

and so $\cdot z e_\theta z \in L$. We now evaluate an expression for this element.

$$\cdot z e_\theta z = \Big(\cdot z \begin{bmatrix}1&0\\0&1\end{bmatrix} + \cdot \sum_{y\neq 0} \theta\begin{pmatrix}1&y\\0&1\end{pmatrix}^{-1} \cdot z \begin{bmatrix}1&y\\0&1\end{bmatrix}\Big) z$$

$$= \cdot 1 + \sum_{\sqrt{y}\in F} \theta(\tfrac{1}{y}) \cdot z \begin{bmatrix}1&y\\0&1\end{bmatrix} + \sum_{\sqrt{y}\notin F} \theta(\tfrac{1}{y}) \cdot xz \begin{bmatrix}1&y\\0&1\end{bmatrix}$$

$$= \cdot 1 + \sum_{y\neq 0} \theta(\tfrac{1}{y}) \cdot z \begin{bmatrix}1&y\\0&1\end{bmatrix} + \sum_{\sqrt{y}\notin F} \theta(\tfrac{1}{y})(\cdot z + \cdot xz) \begin{bmatrix}1&y\\0&1\end{bmatrix}$$

$$= \cdot 1 + \sum_{y} f(y) \cdot z \begin{bmatrix}1&y\\0&1\end{bmatrix} + \sum_{y} g(y)(\cdot z + \cdot xz) \begin{bmatrix}1&y\\0&1\end{bmatrix} \text{ say.}$$

Consider the two functions defined by this equation : f and g .

As in 7.1 we determine (f,σ) and (g,σ) for $\sigma \in \hat{S}$

$$(f, \theta_0^{-1}) = \sum_{y\neq 0} \theta(1/y) = \sum_{y\neq 0} \theta(y) = 1$$

$$(f, (\theta^{x^i})^{-1}) = \sum_{y\neq 0} \theta(\tfrac{1}{y})\, \theta(u^{-2i} y) = \sum_{y\neq 0} \theta(\tfrac{1}{y} + y u^{-2i})$$

$$= \theta(2u^i) + \theta(-2u^i)$$

as $y \to \frac{1}{y} + y u^{-2i}$ has a repeated root at $y = \pm u^i$

i.e. $\frac{1}{y} + y u^{-2i} = \pm 2u^i$

Also $(f, \psi^{-1}) = \sum_{y\neq 0} \theta(\frac{1}{y})\,\theta(ud^2y) = \sum_{y\neq 0}\theta(\frac{1}{y}+ud^2y) = 0$

Now consider g. Let $\mathcal{N} = \{y$ in F_q, y a non-square$\}$

$(g, \theta_0^{-1}) = \sum_{y\in\mathcal{N}} \theta(\frac{1}{y}) = \sum_{y\in\mathcal{N}}\theta(y) = h$ by definition.

$(g, (\theta^{x^{2j}})^{-1}) = \sum_{y\in\mathcal{N}} \theta(\frac{1}{y}+u^{-j}y) = 0$

as $y \to \frac{1}{y}+u^{-j}y$ has a repeated root at only $y = \pm u^{2j}$

which is not a non-square.

$(g, (\theta^{x^{2j+1}})^{-1}) = \sum_{y\in\mathcal{N}} \theta(\frac{1}{y}+u^{-2(2j+1)}y) = \theta(2u^{-2j-1})+\theta(-2u^{-2j-1})$

as $y \to \frac{1}{y}+u^{-2(2j+1)}y$ has a repeated root at $y = \pm u^{2j+1}$

i.e. $\frac{1}{y}+u^{-2(2j+1)}y = \pm 2u^{-2j-1}$

We can now say

$$\sum f(y)\cdot z\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix} = \cdot z e_0 + [\theta(2)+\theta(-2)]\cdot z e_\theta + [\theta(2u)+\theta(-2u)]\cdot z e_{\theta x} + [\theta(2u^2)+\theta(-2u^2)]\cdot z e_{\theta x^2}\ldots$$

$$\sum g(y)(\cdot z+\cdot xz)\begin{bmatrix}1 & y\\ & 1\end{bmatrix} = h(\cdot z+\cdot xz)e_0 + [\theta(2u)+\theta(-2u)](\cdot z+\cdot xz)e_{\theta x}$$
$$+ [\theta(2u^3)+\theta(-2u^3)](\cdot z+\cdot xz)e_{\theta x^3}\ldots\ \ldots + \sum k_\psi(\cdot z+\cdot xz)e_\psi$$

Returning to the expression for $\cdot z e_\theta z$ we have

$$\cdot z e_\theta z = \cdot 1 + \cdot z e_0 + h(\cdot z+\cdot xz)e_0 + \sum_\psi k_\psi(\cdot z+\cdot xz)e_\psi$$
$$+ [\theta(2)+\theta(-2)]\cdot z e_\theta \quad + [\theta(2u^2)+\theta(-2u^2)]\cdot z e_{\theta x^2} \quad \ldots$$
$$+ [\theta(2u)+\theta(-2u)]\cdot xz e_{\theta x} \quad + [\theta(2u^3)+\theta(-2u^3)]\cdot xz e_{\theta x^3}\ldots$$

This cannot belong to L as $\cdot 1 + \cdot z e_0 + h(\cdot z+\cdot xz)e_0 \notin T$

Thus modulo Q, P_0' can have only trivial submodules. $M'/Q \cong M/W \cong k_G$ and $P_0'/M' \cong M$ has a submodule k_G and so P_0'/Q has a two-dimensional trivial submodule. It is not difficult to see that it is generated by $\cdot 1 + \cdot z e_0$, $\quad \cdot 1 + \cdot x$

Modulo Q these elements generate the trivial submodule of P_0'/Q. This is clear by considering the trivial submodule of $\langle \cdot 1, \cdot x, \cdot z e_0, \cdot xz e_0\rangle / T$

Since P_0'/Q can have only trivial submodules in its socle we must actually have that $S(P_0'/Q) \cong k_G \oplus k_G$ as claimed above.

$$(\cdot 1 + \cdot z e_0)\,z = \cdot 1 + \cdot z e_0 + h\sum(\cdot z+\cdot xz)e_\theta + (1+h)\sum(\cdot z+\cdot xz)e_\psi$$

$$(\cdot 1 + \cdot x)\,z = (\cdot z+\cdot xz)e_0 + \sum(\cdot z+\cdot xz)e_\theta + \sum(\cdot z+\cdot xz)e_\psi$$

From these two equations it is clear that P_0'/T has no trivial submodules.This is because no linear combination of $\cdot 1 + \cdot z e_0, \cdot 1 + \cdot x$ can span a trivial submodule modulo T as no linear combination of the Θ, Ψ- components of their images under z is zero.

It is now clear that $s(P_0') = T$, $s^2(P_0') = Q$, $s^3(P_0')/s^2(P_0') \cong k_G \oplus k_G$

Let $I = (\cdot 1 + \cdot z e_0) + (1+h)(\cdot 1 + \cdot x)$ and $J = (\cdot 1 + \cdot z e_0) + h(\cdot 1 + \cdot x)$

Calculation again shows that

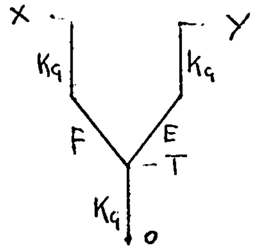
$$I z \in \langle I, \{(\cdot z + \cdot x z) e_\Theta\}, T \rangle \quad = X \text{ say}$$

$$\text{and } J z \in \langle J, \{(\cdot z + \cdot x z) e_\Psi\}, T \rangle \quad = Y \text{ say}$$

It follows that X and Y are submodules uniserial with lattices

$$X = {}_T\left|\begin{array}{l} K_G \\ F \\ K_G \end{array}\right. \qquad Y = {}_T\left|\begin{array}{l} K_G \\ E \\ K_G \end{array}\right.$$

We can now say that $s(P_0'/Q) = s^3(P_0')$ has series



The calculations above establish that

$$\cdot z e_\Theta z \in \langle \cdot z e_\Theta, \cdot x z e_{\Theta x}, \cdot z e_{\Theta x^2}, \cdots, Y \rangle \quad = Y' \text{ say}$$

and similarly $\cdot x z e_{\Theta x} z \in Y'$

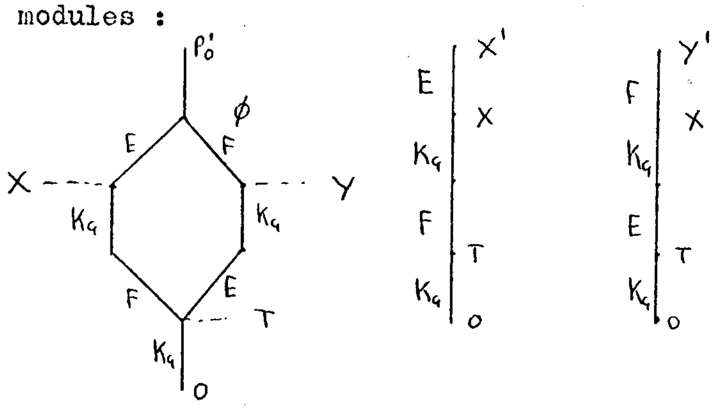
and so on .

Likewise $\cdot z e_\Psi z \in \langle \cdot z e_\Psi, \cdot x z e_{\Psi x}, \cdot z e_{\Psi x^2} \cdots, X \rangle = X'$ say

and $\cdot x z e_{\Psi x} z \in X'$

and so on.

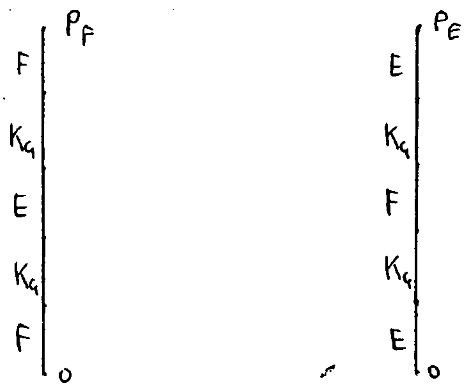
P_0' is self-contragredient and so is $\cong k_G$ modulo its frattini submodule ϕ . P_0' is indecomposable and so $s(P_0') = T$ and as $s^2/s \cong F \oplus E$ we must have that ϕ/s decomposes into at most two summands. It is now routine to establish that the submodules X' and Y' above form a direct sum modulo T and this direct sum is ϕ/s. Also X' and Y' are uniserial of length 4.We can now draw submodule lattices for these modules :



This lattice for the principal projective indecomposable P_0' is in agreement with the results of chapter 6.

Theorem 7.4

The non-principal projective indecomposables P_F and P_E are uniserial with series



Proof

The Cartan matrix says that P_F has composition factors $2F$, $2k_G$, E . The uniserial submodule Y' of P_0' constructed in the previous section has composition length 4 and is a quotient of P_F . Since P_F has a unique irreducible submodule (F) the kernel of the minimal projective presentation of Y' must be $s(P_F) \cong F$ and so the unique series for P_F is as given. An analogous argument applies to P_E .

This concludes the verification of chapter 6 for the second case.

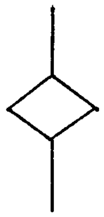
Chapter 8 Miscellaneous Comments

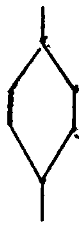
In this chapter are included some remarks which could not conveniently be included in the main argument. Also some of the wider ramifications of these methods are mentioned.

Suppose that in the group G considered previously, the 2-sylow subgroup P is self centralising. Then G has only one block with defect group P. The result of chapter 4 applies and we have that projective indecomposables in blocks of defect 1 have submodule lattices

where I is irreducible.

In blocks of defect 0 of course projectives are irreducible. In the single block of defect group P - 2-B_0 - in case (1) projectives have lattice



and in case (2) have lattice  or 

It follows that if η is the nilpotent radical of kG in case (1) η^3 annihilates all projectives and in case (2) η^5 annihilates them. Since kG is a sum of projectives η^3 and η^5 respectively annihilate kG and so in case (1) $\eta^3 = 0$

and in case (2) $\eta^5 = 0$

We have had occasion to use modules which are projective when restricted to some subgroup, but not themselves projective. Two more examples of the use of these are now given.

(8.1) In section (2) of chapter 5 it is shown that $\Omega^2(K_N)$ has a series

$$\begin{array}{c} \qquad K_N \\ \begin{array}{c|} K_\gamma \\ K_{\gamma^2} \end{array} \oplus \begin{array}{|c} K_{\gamma^2} \\ K_\gamma \end{array} \end{array}$$

We can use this to show that $\Omega^2(K_q)$ has composition length 3 without using 2.21, as follows:

We have that $\begin{pmatrix} \cdot K_\gamma \\ \cdot K_{\gamma^2} \end{pmatrix}^q = F \oplus Pro$

In general $K_N^{\,q} \cong K_q \oplus Pro \oplus X$

so inducing up $\Omega^2(K_N)$ we get a module with series $\begin{array}{c} K_q \oplus X \oplus Pro \\ F \oplus E \oplus Pro \end{array}$

The projective constituents split off (as in 2.10) and we get

$$\Omega^2(K_N)^q = \begin{array}{c} K_q \oplus X \\ F \oplus E \end{array} \oplus Pro = X' \oplus Pro$$

X is a sum of modules of vertex (a).

Now $(F \oplus E)|_{(a)}$ is projective: this is shown in chapter 5.

Now the module X' above has a series $X' \supseteq Y = (X, F \oplus E) \supseteq F \oplus E \supseteq 0$

with $X'/Y \simeq K_q \quad Y/F\oplus E \simeq X$

X is (a)-projective and $F \oplus E$ is projective on (a). This implies that $F \oplus E$ is a summand in Y (see 2.8).

Thus we have $F \oplus E \mid Y \;, \quad Y/F\oplus E \mid X'/F\oplus E$

2.12 now applies and we get $X' \simeq X \oplus X''$

and X'' has series $\begin{array}{c} X'' \cdot K_q \\ \cdot \\ F \oplus E \end{array}$

The argument of chapter 5 (2) now implies that

$$X'' \simeq \Omega^2(K_q)$$

(8.2) Burnside p-complement theorem

Let G be an arbitrary group with p-sylow subgroup P (p odd), P abelian and $N_G(P)$ p-nilpotent. Then G is p-nilpotent. Again by way of demonstration , we prove this using modules projective on a suitable class of subgroups.

Proof of 8.2

The proof is by induction on $|G|$. Let $k = \mathbb{Z}_p$, the field of p elements.

Remark 1 Let G satisfy the hypotheses of 8.2. i.e. $N_G(P)$ is p-nilpotent. Then no two distinct elements of P are conjugate in G. This is well-known and easily proved. It follows that every p-subgroup of G is central in its normaliser. Now suppose by the inductive hypothesis that the result holds for all groups of order less than $|G|$. Firstly suppose $O_{p'}(G) \neq 1$. Then $P/O_{p'}(G) = \overline{P}$ is a p-sylow subgroup of $\overline{G} = G/O_{p'}(G)$. $\overline{N} = N_G(P)/O_{p'}(G)$ $= N_{\overline{G}}(\overline{P})$ and so $\overline{G}$ satisfies the hypotheses of the theorem. Since $|\overline{G}| < |G|$ by induction $\overline{G}$ is p-nilpotent with p-complement $U/O_{p'}(G)$ say. $O_{p'}(G) \in \mathrm{Syl}_p(U)$ and by remark 1, $O_{p'}(G)$ is central in U (and in G). By the Schur-Zassenhaus theorem a p-complement V exists in U and hence is a normal p-complement of U. V char $U \trianglelefteq G$ and so $V \trianglelefteq G$. Thus V is a normal p-complement of G. Thus we may take $O_{p'}(G) = 1$. Let $P = Q \times R$ with Q cyclic of order $p^q \neq 1$. (If no such Q exists, P = 1 and G is p-nilpotent in this case). Let $D \leq Q$ have order p, and $T \leq Q$ have index p.

Remark 2 Let $g,h \in G$. Then $(P \cap P^g)^h \cap Q \neq 1 \Rightarrow g \in N_G(D)$.

Proof of remark 2 Suppose $(P \cap P^g)^h \cap Q \neq 1$. Then $(P \cap P^g)^h \cap Q \geq D$ as D is the unique minimal subgroup of Q. i.e. $D^{h^{-1}} \leq P$. Remark 1 now implies $D^{h^{-1}} = D$. Thus $D \leq P^g$. i.e. $D^{g^{-1}} \leq P$. Remark 1 again implies $D^{g^{-1}} = D$. i.e. $g \in N_G(D)$. Now let $g \notin N_G(D)$. It follows easily from remark 2 and Mackey (2.1) that any $(P \cap P^g)$-projective kG-module is projective when restricted to Q. As we are taking $O_{p'}(G) = 1$, $N_G(D) < G$. As $N_G(P) = C_G(P) \leq C_G(D) = N_G(D)$, $N_G(D)$ satisfies the hypotheses of 8.2 and has order less than G. Thus by induction $N_G(D)$ is p-nilpotent with p-complement W say. Then $(T \times R).W \trianglelefteq N_G(D) = H$ say, and $H/((T \times R).W) \cong C_p$. kC_p and $k(H/((T \times R).W))$ has a 2-dimensional indecomposable module with trivial composition factors. Call it M. We take M to be a kH-module by lifting it modulo $(T \times R).W$. dim(M) = 2 and p is odd.

Theorem 9 of Green (8) implies that vertex(M) = P. We can apply the Green correspondence (2.4) between H and G to modules of vertex P, as $N_G(P) \leq H$. ${k_H}^G \cong k_G \oplus X$ where X is $\mathfrak{X}$-projective. Here $\mathfrak{X} = \{ P \cap P^g \mid g \notin N_G(D) \}$. Thus by the sentence following the proof of remark 2, $X|_Q$ is projective. Now M has a series $\begin{array}{l} \quad M \\ k_H \\ k_H \\ \quad 0 \end{array}$ and so M^G has a series $\begin{array}{c} k_G \oplus X \\ k_G \oplus X \end{array}$ and so also a series $\begin{array}{ll} X & M \\ k_G & M' \\ k_G & M'' \\ X & 0 \end{array}$.

Let $M'/M'' = \widetilde{M}$.

Thus $M^G|_Q \cong \widetilde{M}|_Q \oplus$ projectives. Let Γ be a (H,Q)-transversal of G with $1 \in \Gamma$ i.e. $G = H.\Gamma.Q$. By Mackey (2.1), $M^G|_Q = M|_Q \oplus \sum_{g \in \Gamma \setminus 1} M^g|_{H^g \cap Q}{}^Q$

The construction of M clearly shows that $M|_Q$ is indecomposable and non-projective. Thus $M|_Q$ is a summand of $\widetilde{M}|_Q$ and so $M|_Q = \widetilde{M}|_Q$ as dim(M) = 2 = dim($\widetilde{M}$). Hence $\widetilde{M}$ is non-splitting on Q and so non-splitting on G. Thus $K \neq G$ where $K = \ker(\widetilde{M})$. The argument of 2.13 now implies that G/K is a p-group. Let $S = P \cap K \in Syl_p(K)$. If S = 1, K is already a normal p-complement of G. Take $S \neq 1$. $N_G(P) = C_G(P) \leq C_G(S) = N_G(S)$ by remark 1. $N_G(S) < G$ as $O_p(G) = 1$. By induction $N_G(S)$ is p-nilpotent. Hence $N_K(S)$ is p-nilpotent and by induction K is p-nilpotent with p-complement Y say. Y char $K \triangleleft G \Rightarrow Y \trianglelefteq G$ and so Y is a normal p-complement of G. This completes the proof of 8.2.

(8.3) It is interesting to note that for an arbitrary group and subgroup $X \subseteq Y$ linear sums of Y-modules which are projective on X form an ideal in Green's representation algebra (see Green (9)); this ideal annihilates the ideal of X-projective modules modulo the fully projective ideal. This is because if M is projective on X, N is X-projective

$(M \otimes N)_X = M_X \otimes N_X$ is projective but

$N \mid {N_X}^Y \Rightarrow M \otimes N \mid M \otimes {N_X}^Y = (M|_X \otimes N|_X)^Y$

and so $M \otimes N$ is both X-projective and projective on X and so is projective.

If f,g are the Green correspondences between some group G and subgroup H with classes $\mathfrak{X}$ and $\mathcal{Y}$, then for kG modules of vertex satisfying the conditions of 2.4 which are projective on all $X \in \mathfrak{X}$, we can say M is liftable to R $\Leftrightarrow$ f(M) is liftable to R. This is proved using an argument as in 2.7 or theorem 2 of Green (11).
Also modules projective on the class of subgroups $\mathfrak{X}$ correspond under f and g. The operator Ω also preserves this condition and a "D-presentation" of a module projective on D is just an ordinary projective presentation (see 8.11).

(8.4) Return now to the group G of chapters 5 and 6. The results of those chapters and 2.20 indicate that for many G-representations M and N-modules L of vertex P we can say

$M|_N \simeq f(M) \oplus$ projective and $N^G \simeq g(N) \oplus$ proj $\oplus$ non-princ.

Because of this we can refine the Green isomorphism 2.5 to

$$\frac{(M, M')_G}{(M, M')_{(1),G}} \simeq \frac{(fM, fM')_N}{(fM, fM')_{(1),N}} \qquad \text{in many cases}$$

(of course all modules are P-projective so $(M, M')_{P,G} \simeq (M, M')_G$

We can also define f,g on the module of vertex (a) in B_0 since chapters 5 and 6 again show that for this module $A'|_N = A \oplus \mathrm{Pro}$

A indecomposable of vertex (a) and

$A^G \cong A' \oplus$ projective $\oplus$ non-principal.

The matrices D_0 could also be deduced using this sharper correspondence theorem together with 2.6.

(8.5) Liftings of the module A'.

The module $\widetilde{k_{\langle a\rangle}^P}$ on C has two obvious liftings to R viz. $\widetilde{R_{(a)}^P}$ $\widetilde{R_{(a)}^{-P}}$

These are the only two liftings of $\widetilde{k_{(a)}^P}$ to RC-modules of vertex (a).

By the argument of 2.7 $g(\widetilde{R_{(a)}^P})$, $g(\widetilde{R_{(a)}^{-P}})$

are distinct liftings of A' to RG. They are of vertex (a) and are uniquely determined by these properties.

(8.6) On page 439 of Gorenstein (7) an argument of Glauberman is presented. The characters ρ_i used there are the characters of our representations A_i, A_i^-, of 4.1(1) and the characters χ_i are the characters of our representations $g(A_i)$, $g(A_i^-)$

$$(\rho_i - \rho_i^-)^\tau = \chi_i - \chi_i^- \ , \ \chi_i^- - \chi_i$$

depending on the value of ϵ.

In the case that F and E are odd dimensional

$$\chi(g(\widetilde{R_{(a)}^P})) = \chi_0 + \chi' + \chi'' + \chi \ , \ \chi(g(\widetilde{R_{(a)}^{-P}})) = 2\chi$$

$g(\widetilde{R_{(a)}^{-P}})$ and $g(\widetilde{R_{(a)}^P})$ are projective kernels of each other and so the sum of their characters is projective; also they are modularly equal to A'. These two facts together with the form of D_0, afford this result. That they are projective kernels is clear as there is an exact sequence

$$0 \to R_{(a)} \to R(a) \to R_{(a)}^- \to 0$$

and so $0 \to R_{(a)}^P \to RP \to R_{(a)}^{-P} \to 0$ etc. is exact.

In case F and E have even dimension

$$\chi(g(\widetilde{R_{(a)}^P})) = \chi_0 + \chi \qquad \chi(g(\widetilde{R_{(a)}^{-P}})) = \chi' + \chi''$$

This is proved by the same argument.

The linear combination θ considered in Gorenstein (7) page 439 Ex 1(4) turns out to be simply $\chi(g(\widetilde{R_{(a)}^P})) - \chi(g(\widetilde{R_{(a)}^{-P}}))$ in each case.

(8.7) In the case of a group whose 2-sylow subgroup P is dihedral of order $\geqslant 8$, the centraliser C of the central involution is 2-nilpotent and contains the normaliser of P. The class $\mathfrak{X}$ in general contains elementary abelian subgroups of order 4, together with cyclic subgroups of order 2. This introduces complications and the methods used in this thesis do not obviously overcome them. However the structure of the principal block of such groups is very like that for the case we have discussed and so analogous results may well be obtainable.

There now follow some random remarks on groups with quaternion 2-sylow subgroups.

(8.8) Let G be a group whose 2-sylow subgroup P is quaternion of order 8: $P = \langle a, b,\ a^4 = 1 = b^4,\ a^b = a^{-1}\ \ a^2 = b^2 \rangle \qquad a^2 = J$

$C = C(J)$ is strongly embedded and so under the Green correspondence between G and C, $\mathfrak{X} = \mathfrak{Y} = \{1\}$

$C/(J)$ is a group whose 2-sylow subgroup is elementary abelian of order 4. If k is of characteristic 2 and M is a kC-module $M(J-1)$ is a submodule of M; J is in the kernel of $M(J-1)$ and $M/M(J-1)$.

Let P_i be a projective indecomposable of C in 2-B_0. It is not difficult to see that $P_i(J+1) \simeq P_i/P_i(J-1)$

and is a projective $C/(J)$ module. From this remark it follows that the elements of the Cartan matrix of $B_0(C)$ are twice those for

and so $C_0(C) = \begin{matrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{matrix}\ , \qquad \begin{matrix} 8 & 4 & 4 \\ 4 & 4 & 2 \\ 4 & 2 & 4 \end{matrix}$

We can lift the four ordinary irreducible characters of $C/(J)$ to C. This gives four rows of D_0. The form of C_0 is then seen to determine D_0 and we obtain

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{matrix} \qquad \text{or} \qquad \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

If we could show that irreducibles in $B_0(G)$ and $B_0(C)$ corresponded under f and g , it would follow that the decomposition matrix for $B_0(G)$ was the same as for C. This is because f and g preserve "liftability" as $\mathfrak{X} = \mathcal{Y} = \{1\}$ and commute with the mod 2 map "$-$" (see 2.7).

In the case where modular irreducibles in $B_0(C)$ have odd dimension $\Phi/_S(P_i)$ is very probably indecomposable where P_i is a projective indecomposable in $B_0(C)$. This is because the corresponding section for a projective indecomposable of $N(P)$ is indecomposable. From this the loewy factors of P_i may be determined. The correspondents of modular irreducible kG modules in B_0 may in turn be restricted, using the fact that $\mathfrak{X} = \mathcal{Y} = \{1\}$ For example it can be shown that the number of modular irreducibles in $B_0(G)$ which correspond on C to single-headed or single-footed modules, is 1 or 3.

(8.9) Let $H \leq G$ be groups and f,g the Green correspondences between G and H. Suppose H contains a p-sylow subgroup of G and H intersects its distinct conjugates in subgroups of order prime to p. Then $\mathfrak{X} = \mathcal{Y} \cong \{1\}$ and H contains the normaliser in G of any of its p-sylows. Then we can apply f and g to any non-projective G or H module: let M,N be kG modules.

$$\mathrm{Ext}(M,N) = \frac{(\Omega M, N)}{(\Omega M, N)_{1,G}} \cong \frac{(\Omega f M, f N)}{(\Omega f M, f N)_{1,G}} = \mathrm{Ext}(fM, fN)$$

Let
$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0 \\ 0 \arrow[r] & \Omega M \arrow[r] \arrow[u, "u"] & P \arrow[r] \arrow[u, "?"] & M \arrow[r] \arrow[u, "1_m"] & 0 \end{tikzcd}$$
be exact with P projective.

The map u determines the extension E (see Cartan and Eilenberg (4)). So we can define a map τ: $\mathrm{Ext}(M,N) \to$ extensions of M by N, and by abuse $\tau : \mathrm{Ext}(fM, fN) \to$ " fM fN

It is now not difficult to see that if $t \in \mathrm{Ext}(M,N)$ $t\sigma\tau^{-1}$ is a module isomorphic to $f(t\tau)$ modulo projectives. Thus in this sense the Green mappings σ, f form two sides of a commutative square whose other sides are τ and τ. This is an observation which might well be useful in the kind of work developed in this thesis.

Blocks and Extensions

It is a fact that modules lying in different blocks have only split extensions with each other. It is also true that modules in the same block need not have non-trivial extensions. However, we show that all the irreducible kG modules lying in a block B can be connected by a sequence of non-splitting extensions.

(8.10) Let F,F' be kG irreducible lying in a block B. G and B are arbitrary. Then there exists $F = F_0, F_1, \dots F_n = F'$ with F_i irreducible and non-splitting exact sequences

$$0 \to F_i \to E_i' \to F_{i+1} \to 0 \quad \text{or} \quad 0 \to F_{i+1} \to E_i' \to F_i \to 0$$

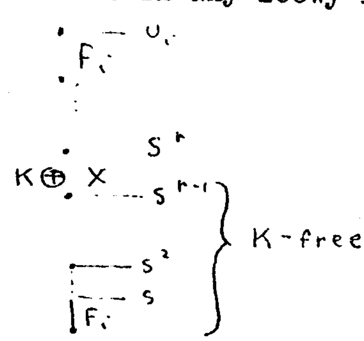
Proof

F and F' are linked in the sense of C.R. 55.1.
Thus there exist $F = E_0, E_1, \dots E_t = F'$ with projective hulls U_i such that U_i, U_{i+1} have a composition factor in common. We show that E_i, E_{i+1} are connected by a sequence of non-splitting exact sequences of the type stated above.

Let U_i U_{i+1} have irreducible composition factor K in common.
$S(U_i) = F_i$; in the "socle" series of U_i defined by

$$\frac{S_n(U_i)}{S_{n-1}(U_i)} = S\left(\frac{U_i}{S_{n-1}(U_i)}\right)$$

choose r so that K occurs in $\frac{S_r}{S_{r-1}}$ but not in any loewy factor with smaller r. Thus U_i has series



Now $\frac{S_{r-1}}{S_{r-2}}$ is completely reducible so $= L_1 \oplus \cdots \oplus L_q$

Then there must exist a non-splitting exact sequence

$$0 \to L_i \to E_i \to K \to 0$$

for some i. For if not K would have only split extensions with the module S_{r-1}/S_{r-2} and so S_r/S_{r-2} would have a series $\begin{array}{l} \cdot S_r \\ K \oplus X \\ \cdot \text{---} S_{r-1} \\ W \\ \cdot \text{---} S_{r-2} \end{array}$

with $W \langle\langle W, K \rangle$, . 2.12 now applies and yields that K is a summand in S_r/S_{r-2} i.e. $K \in \frac{S_{r-1}}{S_{r-2}}$ contrary to choice of r.

Thus there exists a non-splitting $0 \to L_i \to E_i \to K \to 0$ with L_i irreducible.

Now repeat this argument for L_i. Continuing we obtain exact non-splitting series

$$\begin{array}{l} 0 \to L_i \to E_i \to K \to 0 \\ 0 \to L_i^2 \to E_i^2 \to L_i \to 0 \\ \qquad\;\; \vdots \qquad\qquad\qquad\; \vdots \end{array}$$

and eventually $\quad 0 \to F_i \to E_i^v \to L_i^{v-1} \to 0$

Thus K is connected to F_i by exact sequences.

Similarly K is connected to F_{i+1} by exact sequences and so F_i is connected to F_{i+1} by exact sequences. It follows that F is connected to F' by exact sequences. This is what was stated in 8.10.

(8.11) D-presentations

It is shown in Higman (13) that for any group G and subgroup S any kG module M is an S-split quotient of $M_S{}^G$ i.e. there exists

$$M_S{}^G \xrightarrow{\pi} M \to 0$$

with ker π an S-summand of $M_S{}^G$. We call such an epimorphism an S-presentation of M. If S equals (1), S-presentations are simply the projective presentations discussed earlier (in chapter 2).

It is possible to show that we can choose an S-presentation

$$0 \to K \to U \xrightarrow{\pi} M \to 0$$

so that its kernel K is free from S-projective summands. Such an S-presentation is called minimal. Then $U_S = K \oplus K'$ i.e. the kernel K of π is an S-summand of U.

In the earlier discussion it was shown that a projective presentation of $M/\varphi(M)$ can be lifted to one for M and that there is a natural minimal presentation for $M/\varphi(M)$. This is not true as written for S-presentations. However, if we couch it in different language possible generalisations become obvious:

For (1)-presentations, there is a class of kG modules $\mathcal{C}$ completely reducibles - which is R-closed i.e. M/K_1, $M/K_2 \in \mathcal{C} \Rightarrow M/K_1 \cap K_2 \in \mathcal{C}$

Hence, any module M has a unique minimal submodule $\varphi(M)$ such that $M/\varphi(M) \in \mathcal{C}$. Any (1)-presentation of $M/\varphi(M)$ can be lifted to M. Also $\mathcal{C}$-modules have natural minimal (1)-presentations. Thus for a general S-presentation we would like some class $\mathcal{C}_S$ satisfying similar conditions. Some examples of possible solutions to this problem are given below:

$\mathcal{C}_1 = \{M \mid \phi(M) = 0\}$ These are just completely reducible modules.

$\mathcal{C}_2 = \{M \mid \phi(M_S) = 0\}$ These are modules completely reducible on S.

$\mathcal{C}_3 = \{M \mid \phi(M_S) \cap \phi(M) = 0\}$

$\mathcal{C}_4 = \{M \mid$ every maximal submodule of M has an S-complement.$\}$

$\mathcal{C}_5 = \{M \mid$ every submodule of M has an S-complement.$\}$

$\mathcal{C}_6 = \{M \mid$ every submodule of M_S has a complement which is a G-submodule.$\}$

$\mathcal{C}_7 = \{M \mid$ the intersection of those maximal submodules which are H-complemented $= 0\}$

$\mathcal{C}_8$ $\{M \mid$ the intersection of those maximal submodules of M_S which have G-complements $= 0\}$

$\mathcal{C}_9 \quad = \{ M \mid$ the sum of those submodules of M_S containing no proper sub-G-modules = $M \}$

$\mathcal{C}_{10} \quad = \{ M \mid$ the sum of those sub-G-modules of M which are irreducible on S = M $\}$

These classes of course are not unrelated. For example,

$$\mathcal{C}_6 \leq \mathcal{C}_1 \,, \quad \mathcal{C}_2 \leq \mathcal{C}_5$$

(8.12) Consider the results of 2.17. We can say a little more about odd-dimensional modules.

1. The modules $k_N, k_\gamma, k_{\gamma^2}$ have unique liftings $R_N, R_\gamma, R_{\gamma^2}$ to RN.
2. Ω commutes with the "bar" map "—", restriction, and preserves vertices.
3. $g(k_\gamma)|_N = k_\gamma \oplus$ projectives, $\quad g(k_{\gamma^2})|_N = k_{\gamma^2} \oplus$ projectives.

It follows simply from this that for M a kG module of odd dimension in $2\text{-}B_0$

(1) M has a unique lifting M' to R: $\overline{M'} = M$

(2) $M|_N \;=\; f(M) \oplus$ projectives.

Thus these odd-dimensional modules form some of the "many G-representations" of 8.4.

Acknowledgments

Professor J.A. Green has been my supervisor whilst at Warwick. I thank him sincerely for his help. His work is, of course, basic to this whole thesis. More specifically, the arguments of 2.6, 2.19 and Theorem 7.1 are his.

The proof of Theorem 3.1 is adapted from Heller and Reiner (12) (Theorem 2.6), and that of 2.16 from Johnson (14).

I was maintained throughout my period of study at Warwick by a grant from the Carnegie Trust for the Universities of Scotland.

References

(1) Brauer, R. Some applications of the theory of blocks of characters of finite groups I. J. Algebra 1, (1964) 152 - 167

(2) Brauer, R. Some applications of the theory of blocks of characters of finite groups II. J. Algebra 1, (1964) 307 - 334

(3) Brauer, R. Zur Darstellungstheorie der Gruppen endlicher Ordnung II. Math. Z. 72 (1959) 25 - 46

(4) Cartan, H. and Eilenberg, S. Homological Algebra (Princeton, 1956)

(5) Conlon, S. Certain representation algebras. J. Aus. Math. Soc. 5, (1965) 83 - 99

(6) Curtis, C.W. and Reiner, I. Representation Theory of finite groups and associative algebras. (Wiley, New York, 1962) (Referred to as C.R. in the text)

(7) Gorenstein, D. Finite groups. (Harper and Row, New York, 1968)

(8) Green, J.A. On the indecomposable representations of a finite group. Math. Z. 70 (1959) 430 - 445.

(9) Green, J.A. A transfer theorem for modular representations. J. Algebra 1, (1964) 73 - 84.

(10) Green, J.A. Some remarks on defect groups. Math. Z. 107 (1968) 133 - 150.

(11) Green, J.A. A lifting theorem for modular representation. Proc. Roy. Soc. 252 (1959) 135-148.

(12) Heller, A. and Reiner, I. Representations of cyclic groups in rings of integers I. Ann. of Math. 76, (1962) 73 - 92.

(13) Higman, D.G. Modules with a group of operators. Duke Math. J. 21 (1954)

(14) Johnson, D.L. Indecomposable representations of the four-group over fields of characteristic 2. J. London Math. Soc 44 (1969)295-298

(15) Thompson, J.G. Vertices and sources. J. Algebra 6 (1967) 1 - 6.

(16) Brauer, R. Uber Systeme hypercomplexer Zahlen. Math. Z. 30 (1929) 79 - 107

(17) Green, J.A. Relative Module Categories for finite groups. J. Pure Appl. Algebra 2 (1972) 371 - 393

(18) Maclane, S. The uniqueness of the Power Series Representations of certain fields with valuations. Ann. Math. 39 (1938) 370-382

(19) Nagao, H. A proof of Brauer's Theorem on generalised decomposition numbers. Nagoya Math. J. 22 (1963) 73 - 77.